Overview

Modelling

Topics to be covered include:

- How to select the appropriate model complexity
- How to build models for a given plant
- How to describe model errors.
- How to linearise nonlinear models

It also provides a brief introduction to certain commonly used models, including

- State space models
- High order differential and high order difference equation models

The Raison *d'être* for Models

The basic idea of feedback is tremendously compelling. Recall the mould level control problem from Lecture 2. Actually, there are only three ways that a controller could manipulate the valve: open, close or leave it as it is. Nevertheless, we have seen already that the precise way this is done involves subtle trade-offs between conflicting objectives, such as speed of response and sensitivity to measurement noise.

The power of a mathematical model lies in the fact that it can be simulated in hypothetical situations, be subject to states that would be dangerous in reality, and it can be used as a basis for synthesizing controllers.

Model Complexity

In building a model, it is important to bear in mind that all real processes are complex and hence any attempt to build an exact description of the plant is usually an impossible goal. Fortunately, feedback is usually very forgiving and hence, in the context of control system design, one can usually get away with rather simple models, provided they capture the essential features of the problem.

We introduce several terms:

- **Nominal model**. This is an approximate description of the plant used for control system design.
- **Calibration model**. This is a more comprehensive description of the plant. It includes other features not used for control system design but which have a direct bearing on the achieved performance.
- **Model error**. This is the difference between the nominal model and the calibration model. Details of this error may be unknown but various bounds may be available for it.

Building Models

A first possible approach to building a plant model is to postulate a specific model structure and to use what is known as a *black box* approach to modeling. In this approach one varies, either by trial and error or by an algorithm, the model parameters until the dynamic behavior of model and plant match sufficiently well.

An alternative approach for dealing with the modeling problem is to use physical laws (such as conservation of mass, energy and momentum) to construct the model. In this approach one uses the fact that, in any real system, there are *basic phenomenological laws* which determine the relationships between all the signals in the system.

In practice, it is common to combine both black box and phenomenological ideas to building a model.

Control relevant models are often quite simple compared to the true process and usually combine physical reasoning with experimental data.

State Space Models

For continuous time systems

$$\frac{dx}{dt} = f(x(t), u(t), t)$$
$$y(t) = g(x(t), u(t), t)$$

For discrete time systems

$$x[k+1] = f_d(x[k], u[k], k)$$
$$y[k] = g_d(x[k], u[k], k)$$

Linear State Space Models

$$
\begin{aligned}
\frac{dx(t)}{dt} &= \mathbf{A}x(t) + \mathbf{B}u(t) \\
y(t) &= \mathbf{C}x(t) + \mathbf{D}u(t)
\end{aligned}
$$

Example 3.3

Consider the simple electrical network shown in *Figure 3.1*. Assume we want to model the voltage $v(t)$

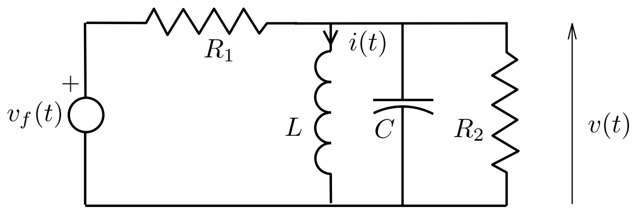


Figure 3.1: *Electrical network. State space model.*

On applying fundamental network laws we obtain the following equations:

$$v(t) = L\frac{di(t)}{dt}$$

$$\frac{v_f(t) - v(t)}{R_1} = i(t) + C\frac{dv(t)}{dt} + \frac{v(t)}{R_2}$$

These equations can be rearranged as follows:

$$\begin{aligned}
\frac{di(t)}{dt} &= \frac{1}{L}v(t) \\
\frac{dv(t)}{dt} &= -\frac{1}{C}i(t) - \left(\frac{1}{R_1C} + \frac{1}{R_2C}\right)v(t) + \frac{1}{R_1C}v_f(t)
\end{aligned}$$

We have a linear state space model with

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\left(\frac{1}{R_1C} + \frac{1}{R_2C}\right) \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{R_1C} \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}; \quad \mathbf{D} = \mathbf{0}$$

Example 3.4

Consider a separately excited d.c. motor. Let $v_a(t)$ denote the armature voltage, $\theta(t)$ the output angle. A simplified schematic diagram of this system is shown in Figure 3.2.

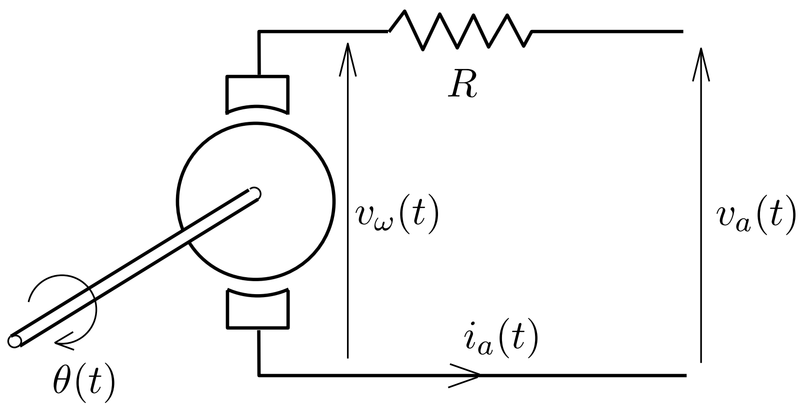


Figure 3.2: *Simplified model of a d.c. motor*

Linearisation

Although almost every real system includes nonlinear features, many systems can be reasonably described, at least within certain operating ranges, by linear models.

Thus consider

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = g(x(t), u(t))$$

Say that $\{x_Q(t), u_Q(t), y_Q(t); t \in \Re^+\}$ is a given set of trajectories that satisfy the above equations, i.e.

$$\dot{x}_Q(t) = f(x_Q(t), u_Q(t)); \qquad x_Q(t_o) \text{ given}$$
$$y_Q(t) = g(x_Q(t), u_Q(t))$$

$$\dot{x}(t) \approx f(x_Q, u_Q) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}} (x(t) - x_Q) + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}} (u(t) - u_Q)$$

$$y(t) \approx g(x_Q, u_Q) + \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}} (x(t) - x_Q) + \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}} (u(t) - u_Q)$$

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t) + \mathbf{E}$$
$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t) + \mathbf{F}$$

$$\mathbf{A} = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}} ; \qquad \mathbf{B} = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}}$$

$$\mathbf{C} = \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}} ; \qquad \mathbf{D} = \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}}$$

$$\mathbf{E} = f(x_Q, u_Q) - \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}} x_Q - \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}} u_Q$$

$$\mathbf{F} = g(x_Q, u_Q) - \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}} x_Q - \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}} u_Q$$

Example 3.6

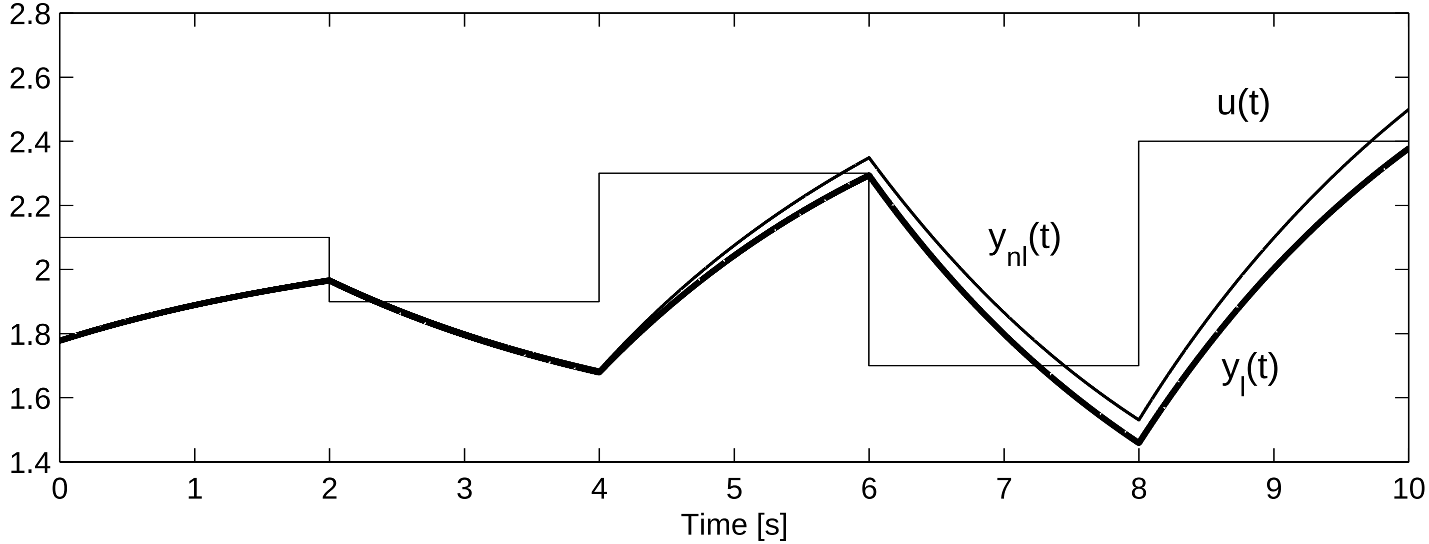
Consider a continuous time system with true model given by

$$\frac{dx(t)}{dt} = f(x(t), u(t)) = -\sqrt{x(t)} + \frac{(u(t))^2}{3}$$

Assume that the input $u(t)$ fluctuates around $u = 2$. Find an operating point with $u_Q = 2$ and a linearized model around it.

$$\frac{d\Delta x(t)}{dt} = -\frac{3}{8}\Delta x(t) + \frac{4}{3}\Delta u(t)$$

Figure 3.4: *Nonlinear system output, $y_{nl}(t)$, and linearised system output, $y_l(t)$, for a square wave input of increasing amplitude, $u(t)$.*



Example 3.7 (Inverted pendulum)

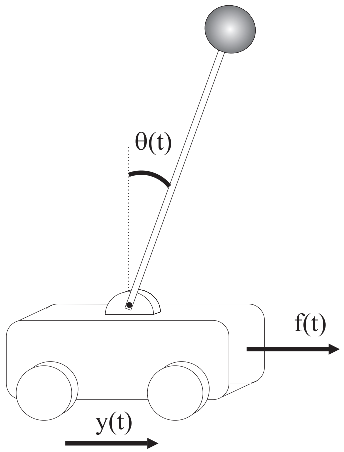


Figure 3.5: *Inverted pendulum*

In Figure 3.5, we have used the following notation:

$y(t)$	-	distance from some reference point
$\theta(t)$	-	angle of pendulum
M	-	mass of cart
m	-	mass of pendulum (assumed concentrated at tip)
ℓ	-	length of pendulum
$f(t)$	-	forces applied to pendulum

Example of an Inverted Pendulum

Application of Newtonian physics to this system leads to the following model:

$$\ddot{y} = \frac{1}{\lambda_m + \sin^2\theta(t)}\left[\frac{f(t)}{m} + \dot{\theta}^2(t)\ell\sin\theta(t) - g\cos\theta(t)\sin\theta(t)\right]$$

$$\ddot{\theta} = \frac{1}{\ell\lambda_m + \sin^2\theta(t)}\left[-\frac{f(t)}{m}\cos\theta(t) + \dot{\theta}^2(t)\ell\sin\theta(t)\cos\theta(t) + (1-\lambda_m)g\sin\theta(t)\right]$$

where $\lambda_m = (M/m)$

This is a linear state space model in which A, B and C are:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{M\ell} & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{M\ell} \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

Summary

- In order to systematically design a controller for a particular system, one needs a formal - though possibly simple - description of the system. Such a description is called a model.
- A model is a set of mathematical equations that are intended to capture the effect of certain system variables on certain other system variables.

- The italicized expressions above should be understood as follows:
  - *Certain system variables*: It is usually neither possible nor necessary to model the effect of every variable on every other variable; one therefore limits oneself to certain subsets. Typical examples include the effect of input on output, the effect of disturbances on output, the effect of a reference signal change on the control signal, or the effect of various unmeasured internal system variables on each other.

- *Capture*: A model is never perfect and it is therefore always associated with a modeling error. The word capture highlights the existence of errors, but does not yet concern itself with the precise definition of their type and effect.
- *Intended*: This word is a reminder that one does not always succeed in finding a model with the desired accuracy and hence some iterative refinement may be needed.
- *Set of mathematical equations*: There are numerous ways of describing the system behavior, such as linear or nonlinear differential or difference equations.

- Models are classified according to properties of the equation they are based on. Examples of classification include:

Model Attribute	*Contrasting Attribute*	*Asserts whether or not …*
Single input Single output	Multiple input multiple output	… the model equations have one input and one output only
Linear	Nonlinear	… the model equations are linear in the system variables
Time varying	Time invariant	… the model parameters are constant
Continuous	Sampled	… model equations describe the behavior at every instant of time, or only in discrete *samples* of time
Input-output	State space	… the model equations rely on functions of input and output variables only, or also include the so called *state variables*.
Lumped parameter	Distributed parameter	… the model equations are ordinary or partial differential equations

- In many situations nonlinear models can be linearised around a user defined operating point.