Overview

Continuous Time Signals

Specific topics to be covered include:

- linear high order differential equation models
- Laplace transforms, which convert linear differential equations to algebraic equations, thus greatly simplifying their study
- methods for assessing the stability of linear dynamic systems
- frequency response.

Linear Continuous Time Models

The linear form of this model is:

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \ldots + a_0 y(t) = b_{n-1}\frac{d^{n-1}}{dt^{n-1}}u(t) + \ldots + b_0 u(t)$$

Introducing the Heaviside, or differential, operator $\rho\langle\circ\rangle$:

$$\rho\langle f(t)\rangle = \rho f(t) \triangleq \frac{df(t)}{dt}$$

$$\rho^n\langle f(t)\rangle = \rho^n f(t) = \rho\left\langle \rho^{n-1}\langle f(t)\rangle\right\rangle = \frac{df^n(t)}{dt^n}$$

We obtain:

$$\rho^n y(t) + a_{n-1}\rho^{n-1}y(t) + \ldots + a_0 y(t) = b_{n-1}\rho^{n-1}u(t) + \ldots + b_0 u(t)$$

Laplace Transforms

The study of differential equations of the type described above is a rich and interesting subject. Of all the methods available for studying linear differential equations, one particularly useful tool is provided by Laplace Transforms.

Definition of the Transform

Consider a continuous time signal $y(t)$; $0 \le t < \infty$. The Laplace transform pair associated with $y(t)$ is defined as

$$\mathcal{L}\left[y(t)\right] = Y(s) = \int_{0^-}^{\infty} e^{-st} y(t) dt$$

$$\mathcal{L}^{-1}\left[y(s)\right] = y(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} e^{st} Y(s) ds$$

A key result concerns the transform of the derivative of a function:

$$\mathcal{L}\left[\frac{dy(t)}{dt}\right] = sY(s) - y(0^-)$$

Table 4.1: *Laplace transform table*

$f(t) \qquad (t \geq 0)$	$\mathcal{L}[f(t)]$	Region of Convergence
1	$\frac{1}{s}$	$\sigma > 0$
$\delta_D(t)$	1	$\lvert\sigma\rvert < \infty$
t	$\frac{1}{s^2}$	$\sigma > 0$
$t^n \qquad n \in \mathbb{Z}^+$	$\frac{n!}{s^{n+1}}$	$\sigma > 0$
$e^{\alpha t} \qquad \alpha \in \mathbb{C}$	$\frac{1}{s-\alpha}$	$\sigma > \Re\{\alpha\}$
$te^{\alpha t} \qquad \alpha \in \mathbb{C}$	$\frac{1}{(s-\alpha)^2}$	$\sigma > \Re\{\alpha\}$
$\cos(\omega_o t)$	$\frac{s}{s^2+\omega_o^2}$	$\sigma > 0$
$\sin(\omega_o t)$	$\frac{\omega_o}{s^2+\omega_o^2}$	$\sigma > 0$
$e^{\alpha t}\sin(\omega_o t + \beta)$	$\frac{(\sin\beta)s + \omega_o^2\cos\beta - \alpha\sin\beta}{(s-\alpha)^2+\omega_o^2}$	$\sigma > \Re\{\alpha\}$
$t\sin(\omega_o t)$	$\frac{2\omega_o s}{(s^2+\omega_o^2)^2}$	$\sigma > 0$
$t\cos(\omega_o t)$	$\frac{s^2-\omega_o^2}{(s^2+\omega_o^2)^2}$	$\sigma > 0$
$\mu(t) - \mu(t-\tau)$	$\frac{1-e^{-s\tau}}{s}$	$\lvert\sigma\rvert < \infty$

Table 4.2: *Laplace transform properties. Note that*

$F_i(s) = \quad [f_i(t)], Y(s) = \quad [y(t)], k \in \{1,2,3,\ldots\}, f_1(t) = f_2)(t) = 0 \; \forall t < 0.$

$f(t)$	$\mathcal{L}[f(t)]$	Names
$\sum_{i=1}^{l} a_i f_i(t)$	$\sum_{i=1}^{l} a_i F_i(s)$	Linear combination
$\frac{dy(t)}{dt}$	$sY(s) - y(0^-)$	Derivative Law
$\frac{d^k y(t)}{dt^k}$	$s^k Y(s) - \sum_{i=1}^{k} s^{k-i} \left.\frac{d^{i-1}y(t)}{dt^{i-1}}\right\rvert_{t=0^-}$	High order derivative
$\int_{0^-}^{t} y(\tau)d\tau$	$\frac{1}{s}Y(s)$	Integral Law
$y(t-\tau)\mu(t-\tau)$	$e^{-s\tau}Y(s)$	Delay
$ty(t)$	$-\frac{dY(s)}{ds}$	
$t^k y(t)$	$(-1)^k \frac{d^k Y(s)}{ds^k}$	
$\int_{0^-}^{t} f_1(\tau)f_2(t-\tau)d\tau$	$F_1(s)F_2(s)$	Convolution
$\lim_{t\to\infty} y(t)$	$\lim_{s\to 0} sY(s)$	Final Value Theorem
$\lim_{t\to 0^+} y(t)$	$\lim_{s\to\infty} sY(s)$	Initial Value Theorem
$f_1(t)f_2(t)$	$\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F_1(\zeta)F_2(s-\zeta)d\zeta$	Time domain product
$e^{at}f_1(t)$	$F_1(s-a)$	Frequency Shift

Transfer Functions

Taking Laplace Transforms converts the differential equation into the following algebraic equation

$$s^n Y(s) + a_{n-1} s^{n-1} Y(s) + ::: + a_0 Y(s)$$
$$= b_{n-1} s^{n-1} U(s) + ::: + b_0 U(s) + f(s; x_o)$$

where
$$Y(s) = G(s)U(s)$$

and
$$G(s) = \frac{B(s)}{A(s)}$$

$G(s)$ is called the *transfer function.*

$$A(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_0$$
$$B(s) = b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_0$$

Transfer Functions for Continuous Time State Space Models

Taking Laplace transform in the state space model equations yields

$$sX(s) - x(0) = \mathbf{A}X(s) + \mathbf{B}U(s)$$
$$Y(s) = \mathbf{C}X(s) + \mathbf{D}U(s)$$

and hence

$$X(s) = (s\mathbf{I} - \mathbf{A})^{-1}x(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s)$$
$$Y(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]U(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}x(0)$$

$$Y(s) = \mathbf{G}(s)U(s)$$
$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

$G(s)$ is the system transfer function.

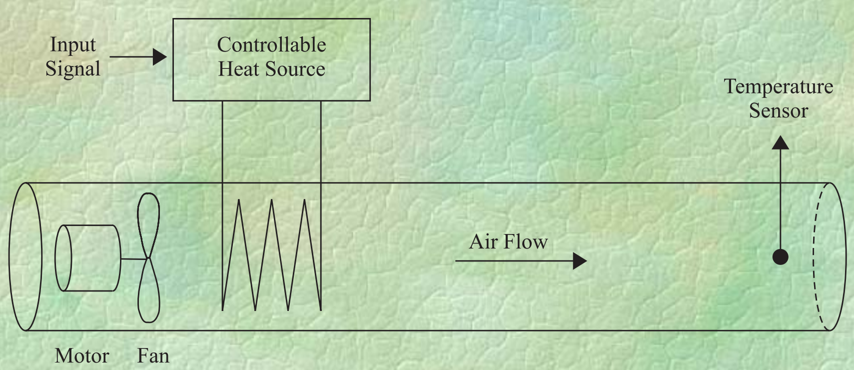
Often practical systems have a time delay between input and output. This is usually associated with the transport of material from one point to another. For example, if there is a conveyor belt or pipe connecting different parts of a plant, then this will invariably introduce a delay.

The transfer function of a pure delay is of the form (see Table 4.2):

$$H(s) = e^{-sT_d}$$

where T_d is the delay (in seconds). T_d will typically vary depending on the transportation speed.

Example 4.4 (Heating system). *As a simple example of a system having a pure time delay consider the heating system shown below.*



The transfer function from input (the voltage applied to the heating element) to the output (the temperature as seen by the thermocouple) is approximately of the form:

$$H(s) = \frac{Ke^{-sT_d}}{(\tau s + 1)}$$

Summary

Transfer functions describe the input-output properties of linear systems in algebraic form.

Stability of Transfer Functions

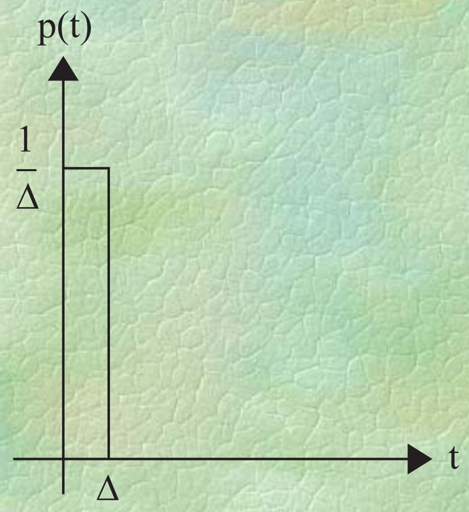
We say that a system is stable if any bounded input produces a bounded output for all bounded initial conditions. In particular, we can use a partial fraction expansion to decompose the total response of a system into the response of each pole taken separately. For continuous-time systems, we then see that stability requires that the poles have strictly negative real parts, i.e., they need to be in the open left half plane (OLHP) of the complex plane $\boxed{s}$. This implies that, for continuous time systems, the stability boundary is the imaginary axis.

Impulse and Step Responses of Continuous-Time Linear Systems

The transfer function of a continuous time system is the Laplace transform of its response to an impulse (Dirac's delta) with zero initial conditions.

The impulse function can be thought of as the limit ($\Delta \to 0$) *of the pulse shown on the next slide.*

Figure 4.2: *Discrete pulse*



Notes on Continuous Time Signals

Steady State Step Response

The steady state response (provided it exists) for a unit step is given by

$$\lim_{t\to\infty} y(t) = y_\infty = \lim_{s\to\infty} sG(s)\frac{1}{s} = G(0)$$

where $G(s)$ is the transfer function of the system.

We define the following indicators:

Steady state value, y_∞: the final value of the step response (this is meaningless if the system has poles in the CRHP).

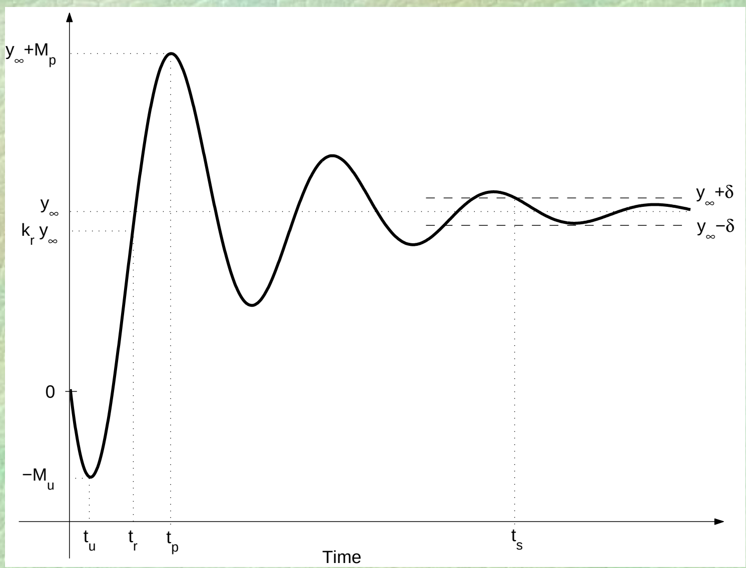
Rise time, t_r: The time elapsed up to the instant at which the step response reaches, for the first time, the value $k_r y_\infty$. The constant k_r varies from author to author, being usually either 0.9 or 1.

Overshoot, M_p: The maximum instantaneous amount by which the step response exceeds its final value. It is usually expressed as a percentage of y_∞

Undershoot, M_u: the (absolute value of the) maximum instantaneous amount by which the step response falls below zero.

Settling time, t_s: the time elapsed until the step response enters (without leaving it afterwards) a specified deviation band, $\pm\delta$, around the final value. This deviation δ, is usually defined as a percentage of y_∞, say 2% to 5%.

Figure 4.3: *Step response indicators*



Notes on Continuous Time Signals

Poles, Zeros and Time Responses

We will consider a general transfer function of the form

$$H(s) = K \frac{\prod_{i=1}^{m}(s-\beta_i)}{\prod_{l=1}^{n}(s-\alpha_l)}$$

$\beta_1, \beta_1, \ldots, \beta_m$ and α_1, α_2, ,,, α_n are the zeros and poles of the transfer function, respectively. The relative degree is $n_r \stackrel{\Delta}{=} n - m$.

Poles

Recall that any scalar rational transfer function can be expanded into a partial fraction expansion, each term of which contains either a single real pole, a complex conjugate pair or multiple combinations with repeated poles.

First Order Pole

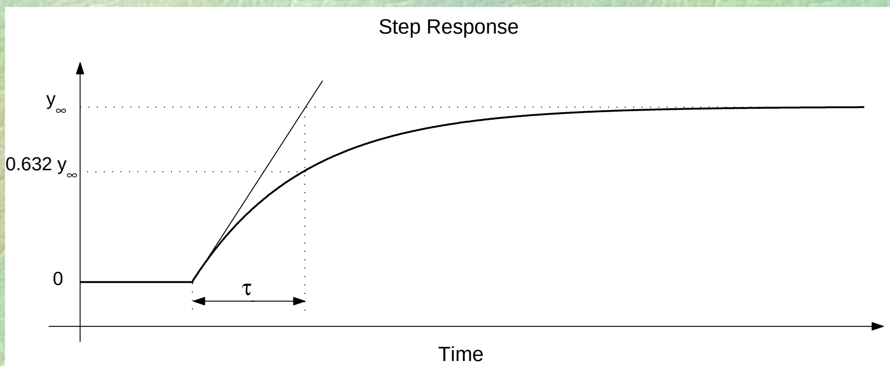
A general first order pole contributes

$$H_1(s) = \frac{K}{\tau s + 1}$$

The response of this system to a unit step can be computed as

$$y(t) = \mathcal{L}^{-1}\left[\frac{K}{s(\tau s + 1)}\right] = \mathcal{L}^{-1}\left[\frac{K}{s} - \frac{K\tau}{\tau s + 1}\right] = K(1 - e^{-\frac{t}{\tau}})$$

Figure 4.4: *Step response of a first order system*



A Complex Conjugate Pair

For the case of a pair of complex conjugate poles, it is customary to study a *canonical second order system* having the transfer function.

$$H(s) = \frac{\omega_n^2}{s^2 + 2\psi\omega_n s + \omega_n^2}$$

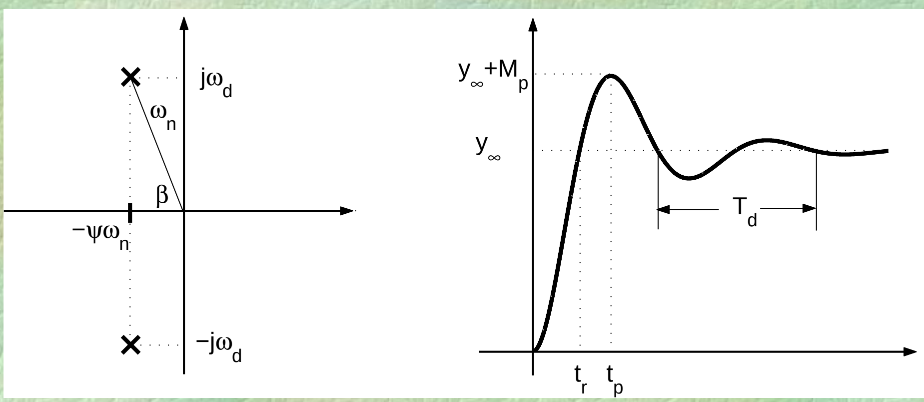
Step Response for Canonical Second Order Transfer Function

$$Y(s) = \frac{1}{s} - \frac{s+\psi\omega_n}{(s+\psi\omega_n)^2+\omega_d^2} - \frac{\psi\omega_n}{(s+\psi\omega_n)^2+\omega_d^2}$$
$$= \frac{1}{s} - \frac{1}{\sqrt{1-\psi^2}}\left[\sqrt{1-\psi^2}\frac{s+\psi\omega_n}{(s+\psi\omega_n)^2+\omega_d^2} - \psi\frac{\omega_d}{(s+\psi\omega_n)^2+\omega_d^2}\right]$$

On applying the inverse Laplace transform we finally obtain

$$y(t) = 1 - \frac{e^{-\psi\omega_n t}}{\sqrt{1-\psi^2}}\sin(\omega_d t + \beta)$$

Figure 4.5: *Pole location and unit step response of a canonical second order system.*



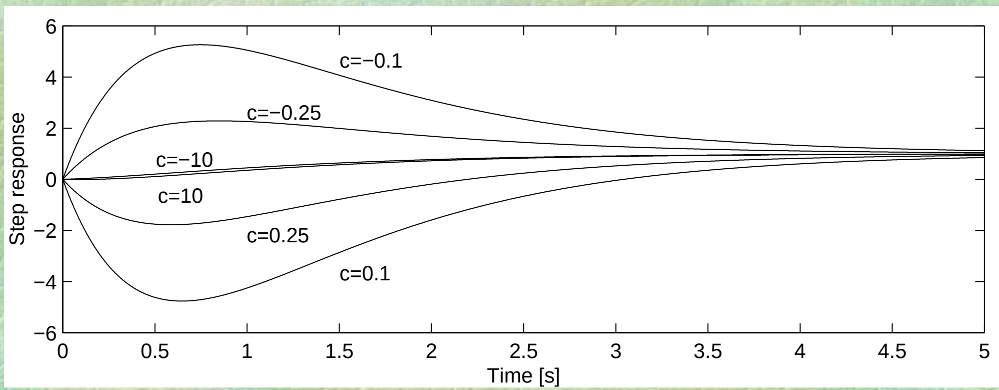
Zeros

The effect that zeros have on the response of a transfer function is a little more subtle than that due to poles. One reason for this is that whilst poles are associated with the states in isolation, zeros rise from additive interactions amongst the states associated with different poles. Moreover, the zeros of a transfer function depend on where the input is applied and how the output is formed as a function of the states.

Consider a system with transfer function given by

$$H(s) = \frac{-s + c}{c(s + 1)(0.5s + 1)}$$

Figure 4.6: *Effect of different zero locations on the step response*



These results can be explained as we show on the next slides.

Analysis of Effect of Zeros on Step Response

A useful result is:

Lemma 4.1: Let $H(s)$ be a strictly proper function of the Laplace variable s with region of convergence $\Re\{s\} > -\alpha$. Denote the corresponding time function by $h(t)$,

$$H(s) = \mathcal{L}\left[h(t)\right]$$

Then, for any z_0 such that $\Re\{z_0\} > -\alpha$, we have

$$\int_0^\infty h(t)e^{-z_0 t}dt = \lim_{s \to z_0} H(s)$$

Non minimum phase zeros and undershoot.
Assume a linear, stable system with transfer function $H(s)$ having unity d.c. gain and a zero at s=c, where $c \in \mathbb{R}^+$. Further assume that the unit step response, $y(t)$, has a settling time t_s (see Figure 4.3) i.e. $1+\delta \geq |y(t)| \geq 1-\delta \, (<<1), \forall t \geq t_s$. Then $y(t)$ exhibits an undershoot M_u which satisfies

$$M_u \geq \frac{1-\delta}{e^{ct_s}-1}$$

The lemma above establishes that, when a system has non minimum phase zeros, there is a trade off between having a fast step response and having small undershoot.

Slow zeros and overshoot. Assume a linear, stable system with transfer function *H*(*s*) having unity d.c. gain and a zero at *s*=*c*, *c*<0. Define *v*(*t*) = 1 - *y*(t), where *y*(*t*) is the unit step response. Further assume that

A-1 The system has dominant pole(s) with real part equal to -*p*, *p*>0

A-2 The zero and the dominant pole are related by

$$\eta \triangleq \left|\frac{c}{p}\right| \ll 1$$

A-3 The value of δ defining the settling time (see Figure 4.3) is chosen such that there exists 0 < *K* which yields

$$|v(t)| < Ke^{-pt} \qquad \forall t \geq t_s$$

Then the step response has an overshoot which is bounded below according to

$$M_p \geq \frac{1}{e^{-ct_s} - 1}\left(1 - \frac{K\eta}{1-\eta}\right)$$

Frequency Response

We next study the system response to a rather special input, namely a sine wave. The reason for doing so is that the response to sine waves also contains rich information about the response to other signals.

Let the transfer function be

$$H(s) = K \frac{\sum_{i=0}^{m} b_i s^i}{s^n + \sum_{k=1}^{n-1} a_k s^k}$$

Then the steady state response to the input $\sin(wt)$ is

$$y(t) = |H(jw)| \sin(wt + \phi(w))$$

where

$$H(j\omega) = |H(j\omega)| e^{j\phi(\omega)}$$

In summary:

A sine wave input forces a sine wave at the output with the same frequency. Moreover, the amplitude of the output sine wave is modified by a factor equal to the magnitude of H(jw) and the phase is shifted by a quantity equal to the phase of H(jw).

Bode Diagrams

Bode diagrams consist of a pair of plots. One of these plots depicts the magnitude of the frequency response as a function of the angular frequency, and the other depicts the angle of the frequency response, also as a function of the angular frequency.

Usually, Bode diagrams are drawn with special axes:

- The abscissa axis is linear in log(w) where the log is base 10. This allows a compact representation of the frequency response along a wide range of frequencies. The unit on this axis is the decade, where a *decade* is the distance between w_1 and $10w_1$ for any value of w_1.

- The magnitude of the frequency response is measured in *decibels* [dB], i.e. in units of $20\log|H(jw)|$. This has several advantages, including good accuracy for small and large values of $|H(jw)|$, facility to build simple approximations for $20\log|H(jw)|$, and the fact that the frequency response of cascade systems can be obtained by adding the individual frequency responses.
- The angle is measured on a linear scale in radians or degrees.

Filtering

In an ideal amplifier, the frequency response would be $H(jw) = K$, constant $\forall w$, i.e. every frequency component would pass through the system with equal gain and no phase.

We define:

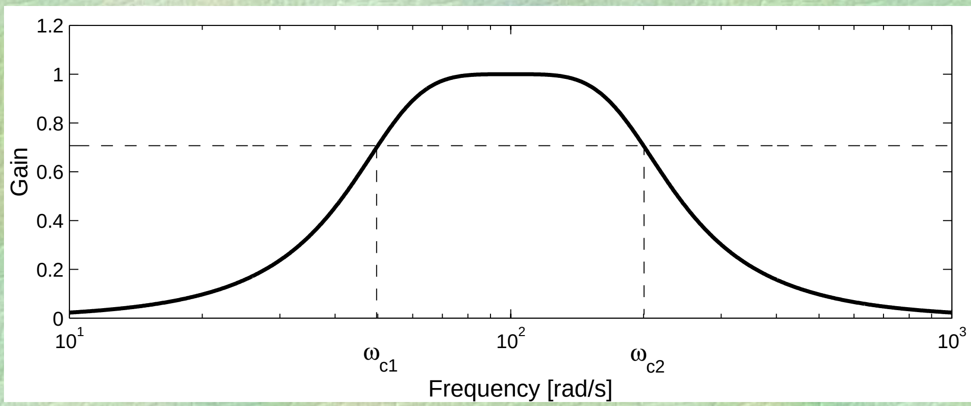
- The *pass band* in which all frequency components pass through the system with approximately the same amplification (or attenuation) and with a phase shift which is approximately proportional to w.

- The *stop band*, in which all frequency components are stopped. In this band $|H(jw)|$ is small compared to the value of $|H(jw)|$ in the pass band.
- The *transition band(s)*, which are intermediate between a pass band and a stop band.

- Cut-off frequency w_c. This is a value of w, such that $|H(jw_c)| = \hat{H}/\sqrt{2}$, where $\hat{H}$ is respectively
  - $|H(0)|$ for low pass filters and band reject filters
  - $|H(\infty)|$ for high pass filters
  - the maximum value of $|H(jw)|$ in the pass band, for band pass filters

- Bandwidth B_w. This is a measure of the frequency width of the pass band (or the reject band). It is defined as $B_w = w_{c2} - w_{c1}$, where $w_{c2} > w_{c1} \geq 0$. In this definition, w_{c1} and w_{c2} are cut-off frequencies on either side of the pass band or reject band (for low pass filters, $w_{c1} = 0$).

Figure 4.8: *Frequency response of a bandpass filter*



Fourier Transform

Definition of the Fourier Transform

$$\mathcal{F}\left[f(t)\right] = F(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} f(t)dt$$

$$\mathcal{F}^{-1}\left[F(j\omega)\right] = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} F(j\omega)d\omega$$

Table 4.3: *Fourier transform table*

$f(t) \qquad \forall t \in \mathbb{R}$	$\mathcal{F}[f(t)]$
1	$2\pi\delta(\omega)$
$\delta_D(t)$	1
$\mu(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\mu(t) - \mu(t - t_o)$	$\frac{1 - e^{-j\omega t_o}}{j\omega}$
$e^{\alpha t}\mu(t) \qquad \Re\{\alpha\} < 0$	$\frac{1}{j\omega - \alpha}$
$te^{\alpha t}\mu(t) \qquad \Re\{\alpha\} < 0$	$\frac{1}{(j\omega - \alpha)^2}$
$e^{-\alpha\lvert t\rvert} \qquad \alpha \in \mathbb{R}^+$	$\frac{2\alpha}{\omega^2 + \alpha^2}$
$\cos(\omega_o t)$	$\pi\left(\delta(\omega - \omega_o) + \delta(\omega - \omega_o)\right)$
$\sin(\omega_o t)$	$j\pi\left(\delta(\omega + \omega_o) - \delta(\omega - \omega_o)\right)$
$\cos(\omega_o t)\mu(t)$	$\pi\left(\delta(\omega - \omega_o) + \delta(\omega - \omega_o)\right) + \frac{j\omega}{-\omega^2 + \omega_o^2}$
$\sin(\omega_o t)\mu(t)$	$j\pi\left(\delta(\omega + \omega_o) - \delta(\omega - \omega_o)\right) + \frac{\omega_o}{-\omega^2 + \omega_o^2}$
$e^{-\alpha t}\cos(\omega_o t)\mu(t) \qquad \alpha \in \mathbb{R}^+$	$\frac{j\omega + \alpha}{(j\omega + \alpha)^2 + \omega_o^2}$
$e^{-\alpha t}\sin(\omega_o t)\mu(t) \qquad \alpha \in \mathbb{R}^+$	$\frac{\omega_o}{(j\omega + \alpha)^2 + \omega_o^2}$

Table 4.4: *Fourier transforms properties. Note that $F_i(jw)$ = $F[f_i(t)]$ and $Y(jw) = F[y(t)]$.*

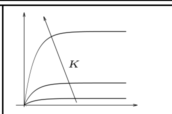
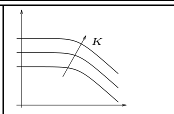
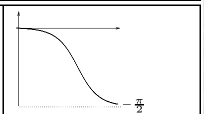
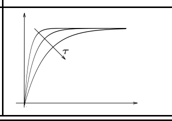
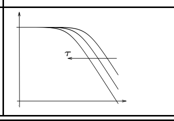
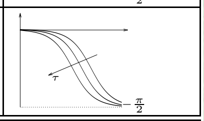
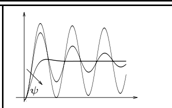
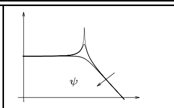
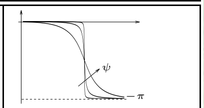
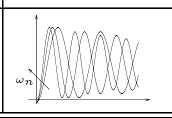
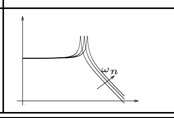
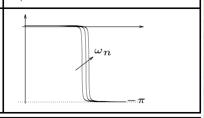
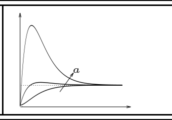
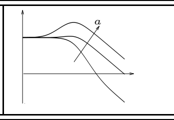
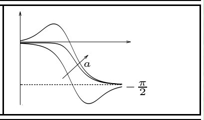
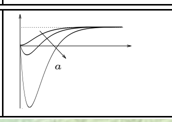
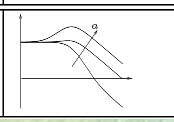
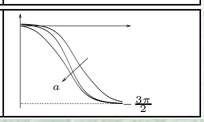
$f(t)$	$\mathcal{F}[f(t)]$	Description
$\sum_{i=1}^{l} a_i f_i(t)$	$\sum_{i=1}^{l} a_i F_i(j\omega)$	Linearity
$\frac{dy(t)}{dt}$	$j\omega Y(j\omega)$	Derivative law
$\frac{d^k y(t)}{dt^k}$	$(j\omega)^k Y(j\omega)$	High order derivative
$\int_{-\infty}^{t} y(\tau)d\tau$	$\frac{1}{j\omega}Y(j\omega) + \pi Y(0)\delta(\omega)$	Integral law
$y(t - \tau)$	$e^{-j\omega\tau}Y(j\omega)$	Delay
$y(at)$	$\frac{1}{\lvert a\rvert}Y\left(j\frac{\omega}{a}\right)$	Time scaling
$y(-t)$	$Y(-j\omega)$	Time reversal
$\int_{-\infty}^{\infty} f_1(\tau)f_2(t - \tau)d\tau$	$F_1(j\omega)F_2(j\omega)$	Convolution
$y(t)\cos(\omega_o t)$	$\frac{1}{2}\left\{Y(j\omega - j\omega_o) + Y(j\omega + j\omega_o)\right\}$	Modulation (cosine)
$y(t)\sin(\omega_o t)$	$\frac{1}{j2}\left\{Y(j\omega - j\omega_o) - Y(j\omega + j\omega_o)\right\}$	Modulation (sine)
$F(t)$	$2\pi f(-j\omega)$	Symmetry
$f_1(t)f_2(t)$	$\frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F_1(\zeta)F_2(s - \zeta)d\zeta$	Time domain product
$e^{at}f_1(t)$	$F_1(j\omega - a)$	Frequency shift

A useful result: *Parseval's Theorem*

Theorem 4.1: *Let F(jw) and G(jw) denote the Fourier transform of f(t) and g(t) respectively. Then*

$$\int_{-\infty}^{\infty} f(t)g(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)G(-j\omega)\,d\omega$$

Table 4.5: *System models and influence of parameter variations*

System	Parameter	Step response	Bode (gain)	Bode(phase)
$\frac{K}{\tau s+1}$	K			
	τ			
$\frac{\omega_n^2}{s^2+2\psi\omega_n s+\omega^2}$	ψ			
	ω_n			
$\frac{as+1}{(s+1)^2}$	a			
$\frac{-as+1}{(s+1)^2}$	a			

Summary

- There are two key approaches to linear dynamic models:
  - the, so-called, time domain, and
  - the so-called, frequency domain
- Although these two approaches are largely equivalent, they each have their own particular advantages and it is therefore important to have a good grasp of each.

- In the time domain,
  - systems are modeled by differential equations
  - systems are characterized by the evolution of their variables (output etc.) in time
  - the evolution of variables in time is computed by solving differential equations

- In the frequency domain,
  - modeling exploits the key linear system property that the steady state response to a sinusoid is again a sinusoid of the same frequency; the system only changes amplitude and phase of the input in a fashion uniquely determined by the system at that frequency,
  - systems are modeled by transfer functions, which capture this impact as a function of frequency.

- With respect to the important characteristic of stability, a continuous time system is
  - stable if and only if the real parts of all poles are strictly negative
  - marginally stable if at least one pole is strictly imaginary and no pole has strictly positive real part
  - unstable if the real part of at least one pole is strictly positive
  - non-minimum phase if the real part of at least one zero is strictly positive.