Overview

Analysis of SISO Control Loops

Topics to be covered

For a given controller and plant connected in feedback we ask and answer the following questions:

y   Is the loop stable?

y   What are the sensitivities to various disturbances?

y   What is the impact of linear modeling errors?

y   How do small nonlinearities impact on the loop?

We recall several analysis tools; specifically

y   Root locus

Feedback Structures

We will see that feedback can have many desirable properties such as the capacity to reduce the effect of disturbances, to decrease sensitivity to model errors or to stabilize an unstable system. We will also see, however, that ill-applied feedback can make a previously stable system unstable, add oscillatory behavior into a previously smooth response or result in high sensitivity to measurement noise.

Figure 5.1: *Simple feedback control system*
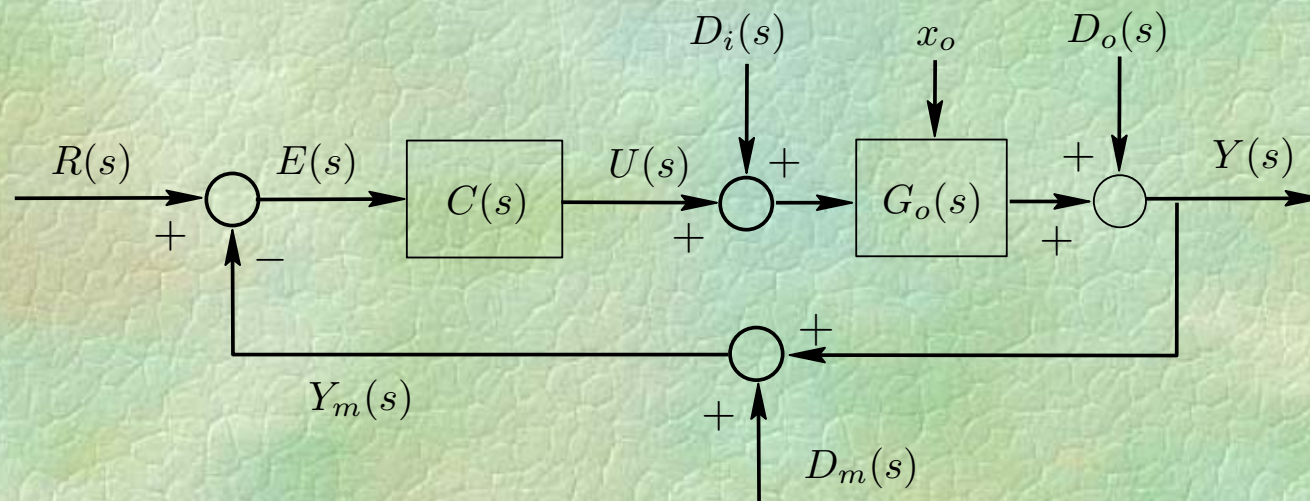
In the loop shown in Figure 5.1 we use transfer functions and Laplace transforms to describe the relationships between signals in the loop. In particular, $C(s)$ and $G_0(s)$ denote the transfer functions of the controller and the nominal plant model respectively, which can be represented in fractional form as:

$$C(s) = \frac{P(s)}{L(s)}$$

$$G_o(s) = \frac{B_o(s)}{A_o(s)}$$

Link to Characteristic Equation

Lemma 5.1 (Nominal internal stability)

Consider the nominal closed loop depicted in Figure 5.2. Then the nominal closed loop is internally stable if and only if the roots of the nominal closed loop characteristic equation

$$A_o(s)L(s) + B_o(s)P(s) = 0$$

all lie in the open left half plane. We call $A_0L + B_0P$ the nominal closed-loop characteristic polynomial.

Stability and Polynomial Analysis

Consider a polynomial of the following form:

$$p(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$$

The problem to be studied deals with the question of whether that polynomial has any root with nonnegative real part. Obviously, this equation can be answered by *computing the n roots* of *p(s)*. However, in many applications it is of special interest to study the interplay between the location of the roots and certain polynomial coefficients.

Root Locus (RL)

A classical tool used to study stability of equations of the type given above is root locus. The root locus approach can be used to examine the location of the roots of the characteristic polynomial as one parameter is varied.

Consider the following equation

$$1 + \lambda F(s) = 0 \qquad \text{where} \qquad F(s) = \frac{M(s)}{D(s)}$$

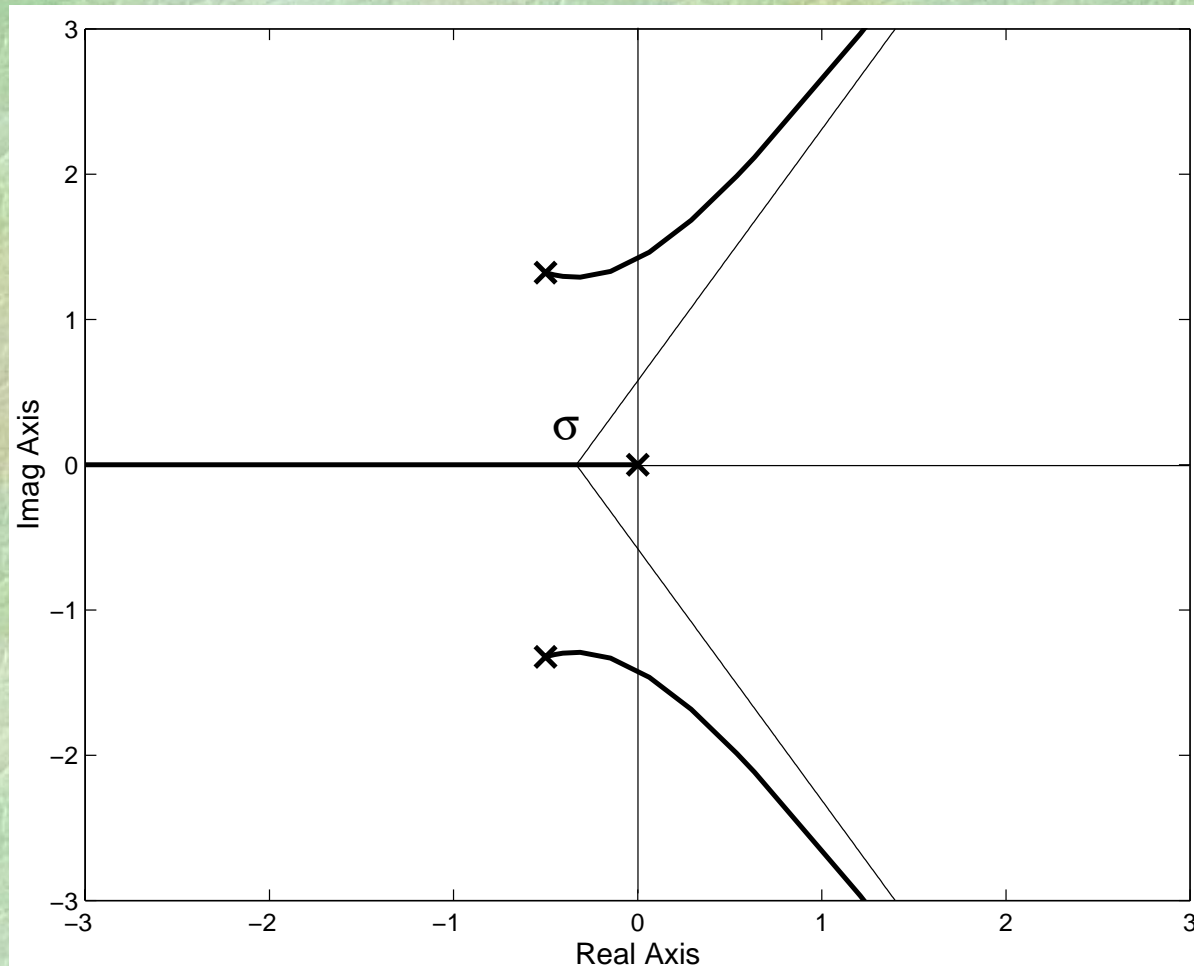with $\lambda \geq 0$ and M, N have degree m, n respectively.

Example

Consider a plant with transfer function $G_0(s)$ and a feedback controller with transfer function $C(s)$, where

$$G_o(s) = \frac{1}{(s-1)(s+2)} \quad \text{and} \quad C(s) = 4\frac{s+\alpha}{s}$$

We want to know how the location of the closed loop poles change for α moving in 6^{+}.

Figure 5.3: *Locus for the closed loop poles when the controller zero varies*

Nominal Stability using Frequency Response

A classical and lasting tool that can be used to assess the stability of a feedback loop is Nyquist stability theory. In this approach, stability of the closed loop is predicted using the open loop frequency response of the system. This is achieved by plotting a polar diagram of the product $G_0(s)C(s)$ and then counting the number of encirclements of the (-1,0) point. We show how this works below.

Final Result

Theorem 5.1:

If a proper open loop transfer function $G_0(s)C(s)$ has P poles in the open RHP, and none on the imaginary axis, then the closed loop has Z poles in the open RHP if and only if the polar plot $G_0(s_w)C(s_w)$ encircles the point (-1,0) clockwise N=Z-P times.

Discussion

- y If the system is open loop stable, then for the closed loop to be internally stable it is necessary and sufficient that no unstable cancellations occur and that the Nyquist plot of $G_0(s)C(s)$ *does not encircle the point* (-1,0).

- y If the system is open loop unstable, with P poles in the open RHP, then for the closed loop to be internally stable it is necessary and sufficient that no unstable cancellations occur and that the Nyquist plot of $G_0(s)C(s)$ *encircles the point* (-1,0) *P times counterclockwise.*

- y If the Nyquist plot of $G_0(s)C(s)$ passes exactly through the point (-1,0), there exists an $w_0 \in 6$ such that $F(jw_0) = 0$, i.e. the closed loop has poles located exactly on the imaginary axis. This situation is known as a *critical stability condition*.

Relative Stability: Stability margins and Sensitivity Peaks

In control system design, one often needs to go beyond the issue of closed loop stability. In particular, it is usually desirable to obtain some quantitative measures of how far from instability the nominal loop is, i.e. to quantify relative stability. This is achieved by introducing measures which describe the distance from the nominal open loop frequency response to the critical stability point (-1,0).

Figure 5.7: *Stability margins and sensitivity peak*

$$|a|$$

$$-1 \qquad \phi$$

$$\omega = \omega_p$$

$$G_o(j\omega)C(j\omega)$$

$$(a)$$
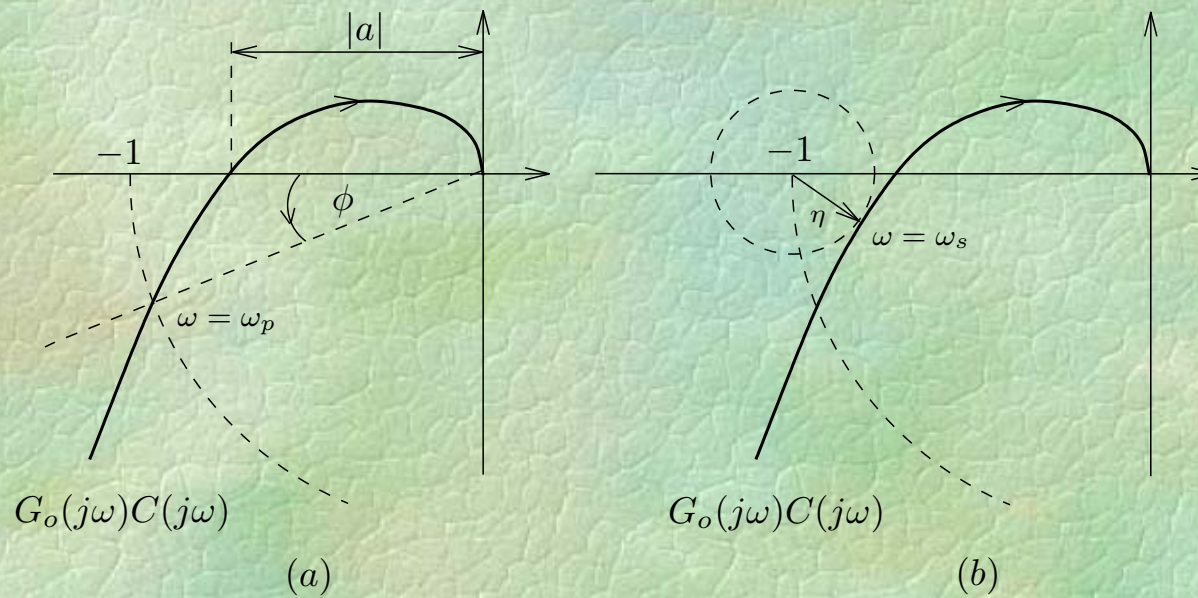
Gain and Phase Margins

$$-1 \qquad \eta \qquad \omega = \omega_s$$

$$G_o(j\omega)C(j\omega)$$

$$(b)$$

Peak Sensitivity

(a) The gain margin, M_g, and the phase margin M_f are defined as follows (see Figure 5.7):
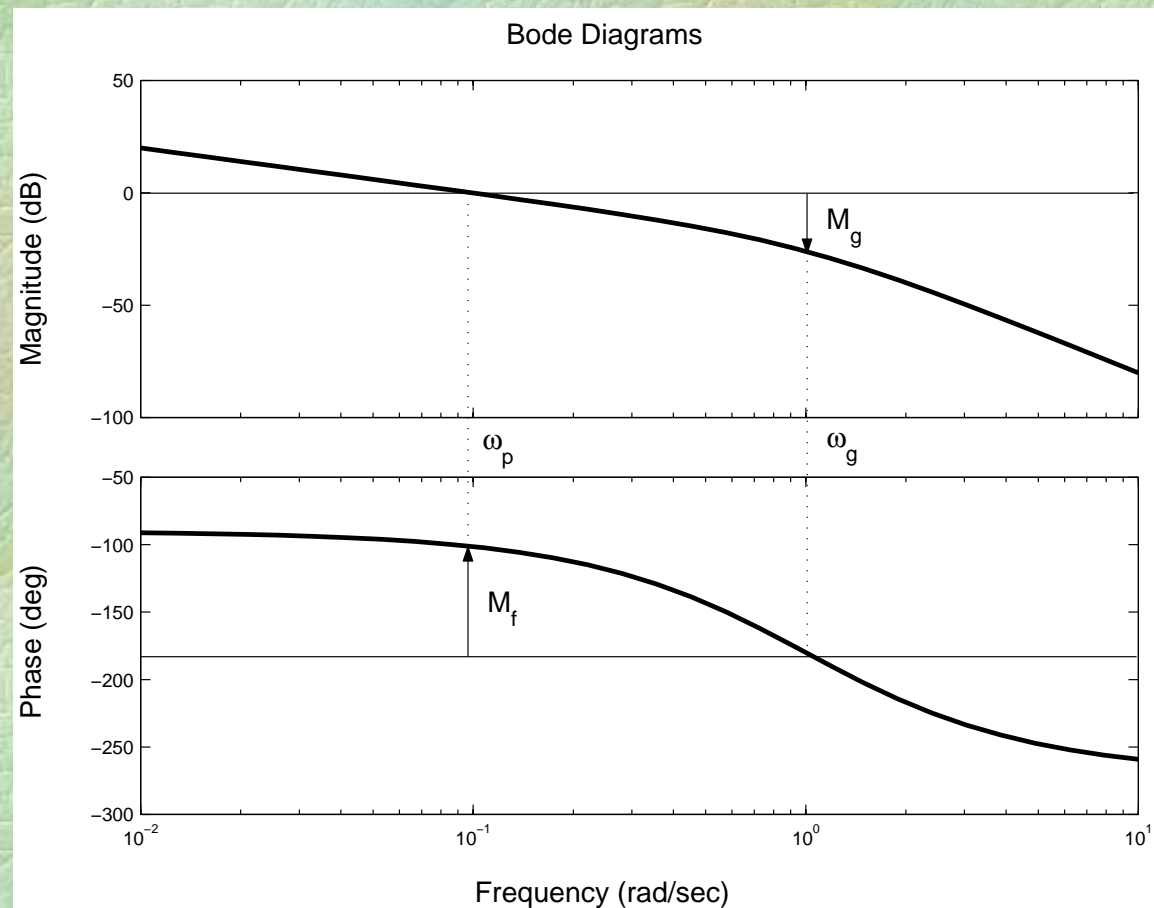
$$M_g \triangleq -20\log_{10}(|a|)$$

$$M_f \triangleq \phi$$

(b) Peak sensitivity:

Since $S_0 = \dfrac{1}{1+G_0C}$, then S_0 is a maximum at the frequency where $G_0(jw)C(jw)$ is closest to the point -1. The peak sensitivity is thus $^1/_\eta$ - (see Figure 5.7).

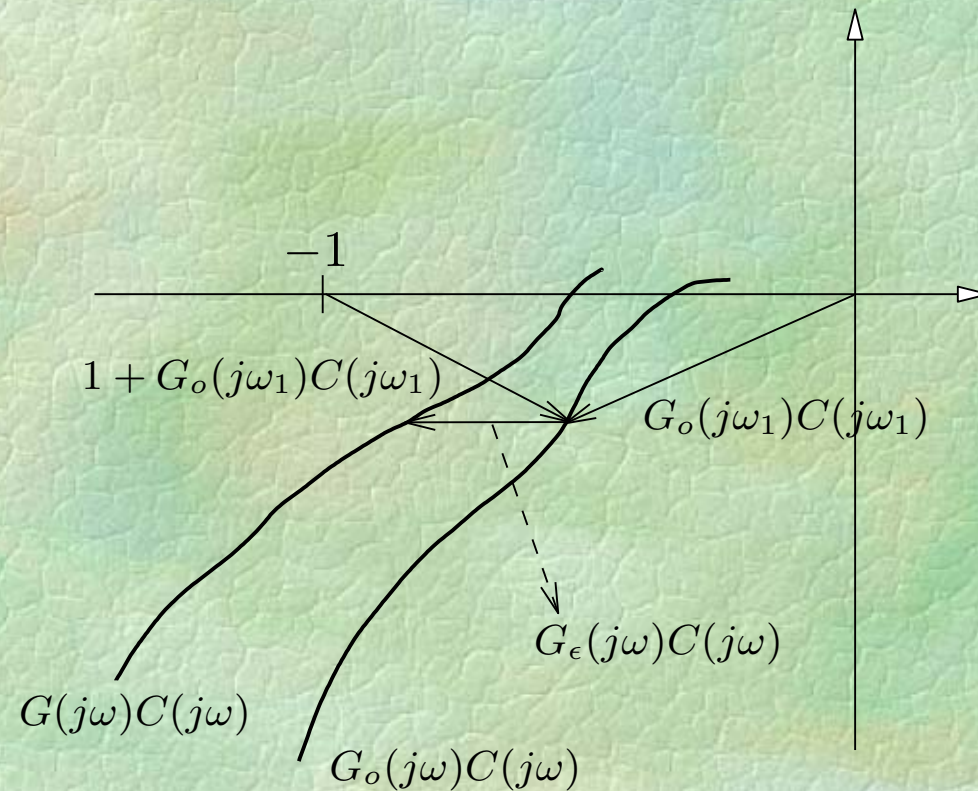## Figure 5.8: *Stability margins in Bode diagrams*

Robustness

So far, we have only considered the effect that the controller has on the nominal closed loop formed with the nominal model for the plant. However, in practice, we are usually interested, not only in this nominal performance, but also the true performance achieved when the controller is applied to the true plant. This is the so called "Robustness" issue. We will show below that the nominal sensitivities do indeed tell us something about the true or achieved sensitivities.

Robust Stability

We are concerned with the case where the nominal model and the true plant differ. It is then necessary that, in addition to nominal stability, we check that stability is retained when the true plant is controlled by the same controller. We call this property *robust stability*.

Proof: *Consider the Nyquist plot for the nominal and the true loop*

-1

$1 + G_o(j\omega_1)C(j\omega_1)$

$G_o(j\omega_1)C(j\omega_1)$

$G_\epsilon(j\omega)C(j\omega)$

$G(j\omega)C(j\omega)$

$G_o(j\omega)C(j\omega)$

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Summary

- y This chapter introduced the fundamentals of SISO feedback control loop analysis.

- y Feedback introduces a cyclical dependence between controller and system:
  - x the controller action affects the systems outputs,
  - x and the system outputs affect the controller action.

y Well designed, feedback can

    x make an unstable system stable;

    x increase the response speed;

    x decrease the effects of disturbances;

    x decrease the effects of system parameter uncertainties, and more.