Overview

Analysis of
continuous-time
SISO Control Loops

Topics to be recalled

For a given controller and plant connected in feedback we ask and answer the following questions:

- y Is the loop stable?
- y What are the sensitivities to various disturbances?
- y What is the impact of linear modeling errors?
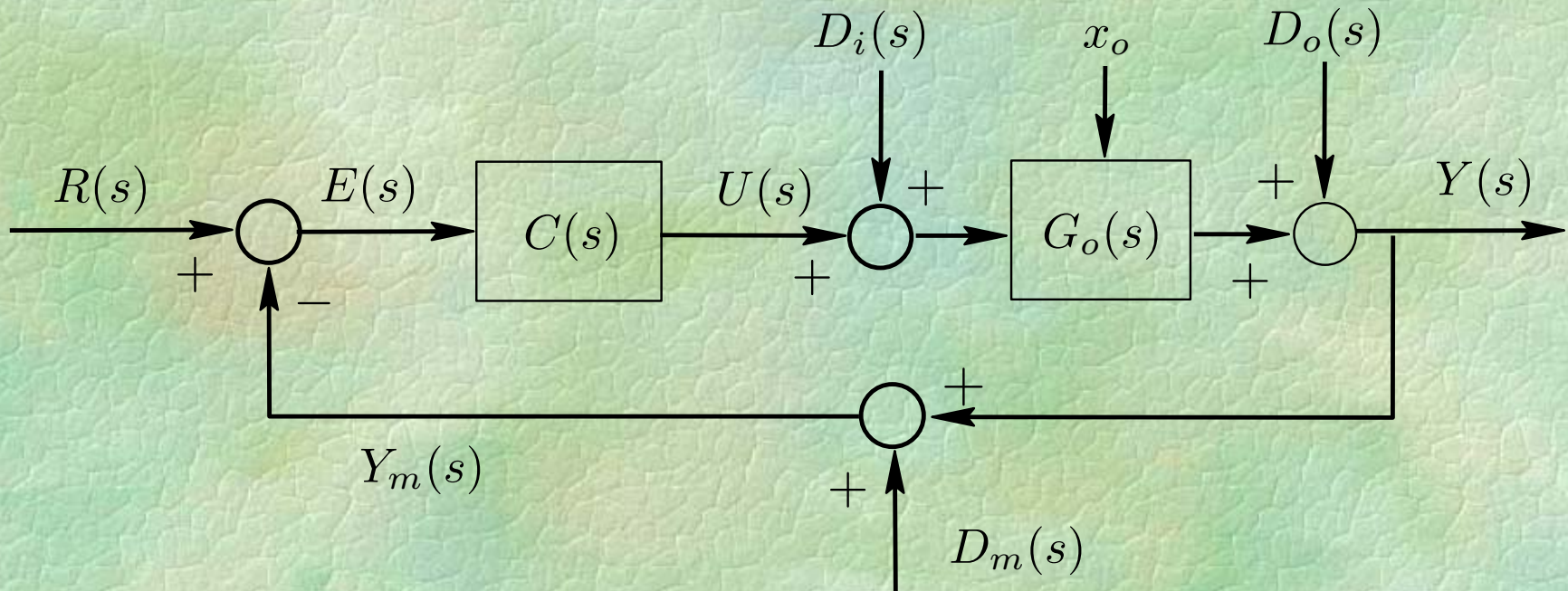- y How do small nonlinearities impact on the loop?
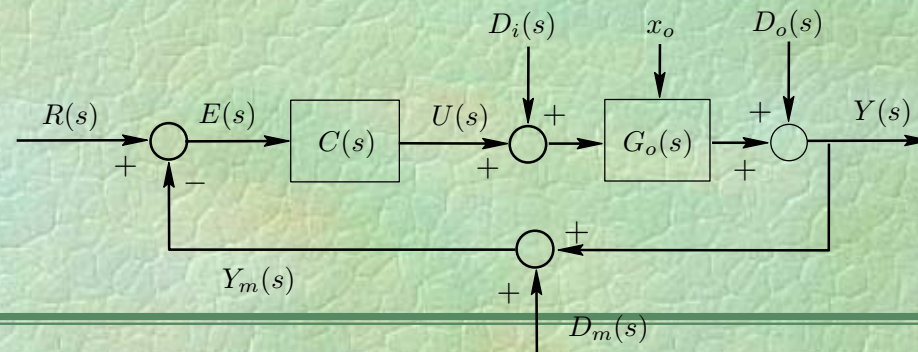
We recall several analysis tools; specifically

- y Root locus

Feedback Structures

We will see that feedback can have many desirable properties such as the capacity to reduce the effect of disturbances, to decrease sensitivity to model errors or to stabilize an unstable system. We will also see, however, that ill-applied feedback can make a previously stable system unstable, add oscillatory behavior into a previously smooth response or result in high sensitivity to measurement noise.

Figure 5.1: *Simple feedback control system (continuous-time)*

In the loop shown in **Figure 5.1** we use transfer functions and Laplace transforms to describe the relationships between signals in the loop. In particular, $C(s)$ and $G_0(s)$ denote the transfer functions of the controller and the nominal plant model respectively, which can be represented in fractional form as:

$$C(s) = \frac{P(s)}{L(s)}$$

$$G_o(s) = \frac{B_o(s)}{A_o(s)}$$

Link to Characteristic Equation

Lemma 5.1 (Nominal internal stability)

Consider the nominal closed loop depicted in Figure 5.1. Then the nominal closed loop is internally stable if and only if the roots of the nominal closed loop characteristic equation

$$A_o(s)L(s) + B_o(s)P(s) = 0$$

all lie in the open left half plane. We call $A_0L + B_0P$ the nominal closed-loop characteristic polynomial.

Stability and Polynomial Analysis

Consider a polynomial of the following form:

$$p(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$$

The problem to be studied deals with the question of whether that polynomial has any root with nonnegative real part. Obviously, this equation can be answered by *computing the n roots* of $p(s)$. However, in many applications it is of special interest to study the interplay between the location of the roots and certain polynomial coefficients.

Root Locus (RL)

A classical tool used to study stability of equations of the type given above is root locus. The root locus approach can be used to examine the location of the roots of the characteristic polynomial as one parameter is varied.

Consider the following equation

$$1 + \lambda F(s) = 0 \qquad \text{where} \qquad F(s) = \frac{M(s)}{D(s)}$$

with $\lambda \geq 0$ and M, N have degree m, n respectively.

Nominal Stability using Frequency Response

A classical and lasting tool that can be used to assess the stability of a feedback loop is Nyquist stability theory. In this approach, stability of the closed loop is predicted using the open loop frequency response of the system. This is achieved by plotting a polar diagram of the product $G_0(s)C(s)$ and then counting the number of encirclements of the (-1,0) point. We show how this works below.
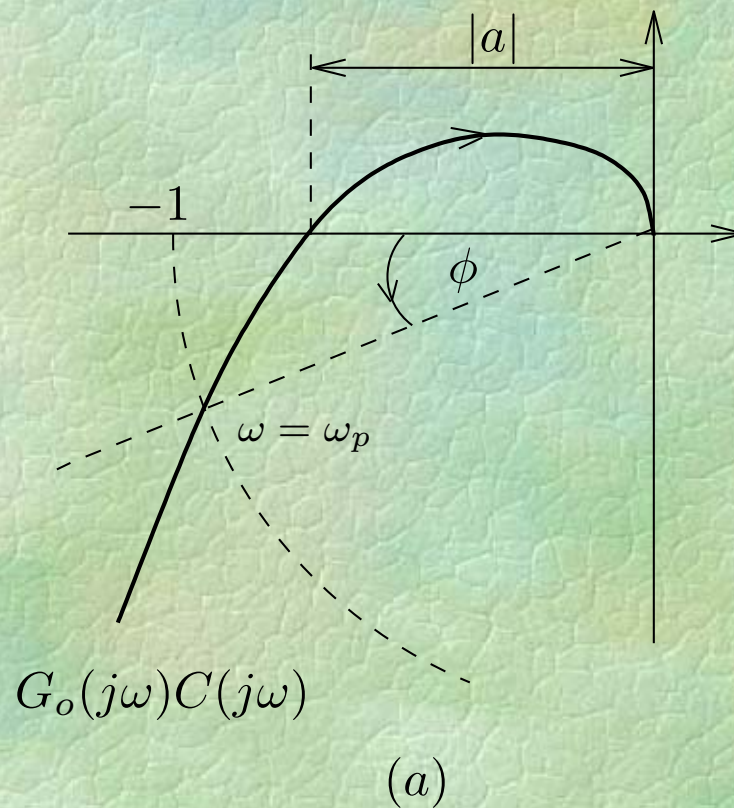
Discussion

y If the system is open loop stable, then for the closed loop to be internally stable it is necessary and sufficient that no unstable cancellations occur and that the Nyquist plot of $G_0(s)C(s)$ *does not encircle the point* (-1,0).

Relative Stability: Stability margins

In control system design, one often needs to go beyond the issue of closed loop stability. In particular, it is usually desirable to obtain some quantitative measures of how far from instability the nominal loop is, i.e. to quantify relative stability. This is achieved by introducing measures which describe the distance from the nominal open loop frequency response to the critical stability point (-1,0).
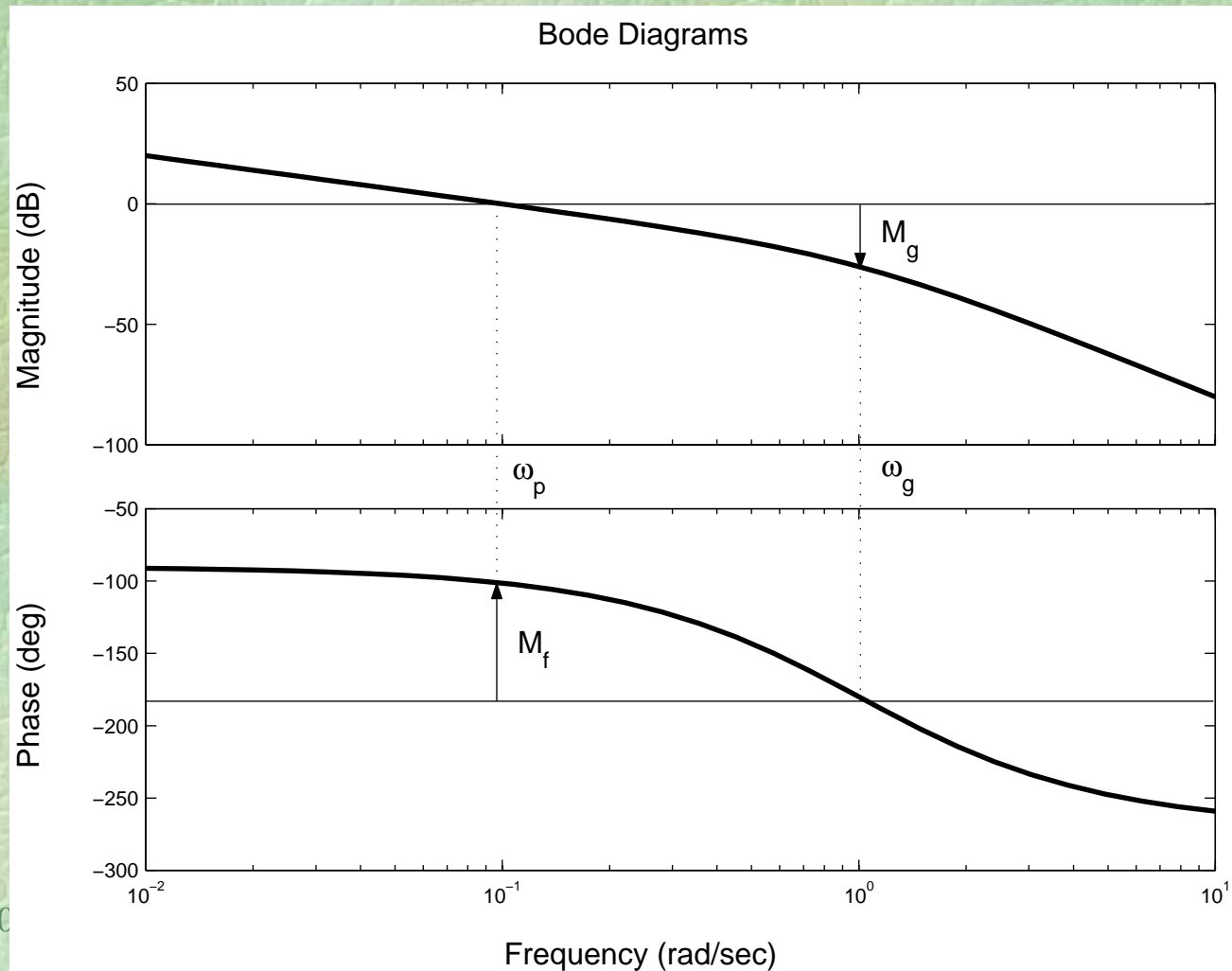
Figure 5.7: *Stability margins*

(a)

Gain and Phase Margins

y   The gain margin, M_g, and the phase margin M_f are defined as follows (see Figure 5.7):

$$M_g \triangleq -20 \log_{10}(|a|)$$

$$M_f \triangleq \phi$$

Figure 5.8: *Stability margins in Bode diagrams*

Robustness

So far, we have only considered the effect that the controller has on the nominal closed loop formed with the nominal model for the plant. However, in practice, we are usually interested, not only in this nominal performance, but also the true performance achieved when the controller is applied to the true plant. This is the so called "Robustness" issue. We will show below that the nominal sensitivities do indeed tell us something about the true or achieved sensitivities.