

## Lecture 4

### Prediction Error Methods (PEM) (Ch. 7)

## The Least-Squares Method

- Chapter 4: The least squares method applied to static (deterministic) linear regression models ( $\varphi(t)$  deterministic).
- What happens when we consider dynamic models?

$$A(q^{-1}, \boldsymbol{\theta})y(t) = B(q^{-1}, \boldsymbol{\theta})u(t) + e(t)$$
$$\Rightarrow y(t) = \boldsymbol{\varphi}^T(t)\boldsymbol{\theta} + e(t)$$

where

$$\boldsymbol{\varphi}(t) = [-y(t-1) \dots -y(t-n_a) \ u(t-1) \dots u(t-n_b)]^T$$
$$\boldsymbol{\theta} = [a_1 \dots a_{n_a} \ b_1 \dots b_{n_b}]^T$$

Properties of the least squares estimate

$$\hat{\boldsymbol{\theta}}_{LS} = \left( \frac{1}{N} \sum_{t=1}^N \boldsymbol{\varphi}(t)\boldsymbol{\varphi}^T(t) \right)^{-1} \frac{1}{N} \sum_{t=1}^N \boldsymbol{\varphi}(t)y(t)$$

**Properties:** Assume that the true system can be described as

$$y(t) = \boldsymbol{\varphi}^T(t)\boldsymbol{\theta}_0 + v(t)$$

**Results:** The estimate  $\hat{\boldsymbol{\theta}}_{LS}$  will be consistent ( $\hat{\boldsymbol{\theta}} \rightarrow \boldsymbol{\theta}_0$  as  $N \rightarrow \infty$ ) if

- (i)  $E\boldsymbol{\varphi}(t)\boldsymbol{\varphi}^T(t)$  is nonsingular.
- (ii)  $E\boldsymbol{\varphi}(t)v(t) = \mathbf{0}$ .

The first condition will be satisfied in most cases. A few exceptions

- The input is not persistently exciting of order  $n_b$ .
- The data is noise-free  $v(t) \equiv 0$  and the model order is chosen too high (which implies that  $A(q^{-1})$  and  $B(q^{-1})$  have common factors).
- The system operates under feedback with a low order regulator.

The second condition is in most cases *not* satisfied. A notable exception is when  $e(t)$  is white noise.

## Modifications of the Least-Squares Method

To relax the second constraint, we will in the following examine two different ways to modify the least-squares method:

- (i) Prediction error methods. Model the noise as well!
- (ii) The instrumental variables methods (IV methods) – modifying the normal equations associated with the least-squares estimate.

## Prediction Error Methods (PEM)

Idea:

- Model the noise as well  $\Rightarrow$  stochastic models, *i.e.*, the outputs from the models are not deterministic.
- Minimize the prediction errors  $\varepsilon(t, \boldsymbol{\theta}) = y(t) - \hat{y}(t|t-1, \boldsymbol{\theta})$ . The least-squares method is a special case of this approach; consider the prediction error

$$\varepsilon(t, \boldsymbol{\theta}) = y(t) - \hat{y}(t|t-1, \boldsymbol{\theta}) = y(t) - \boldsymbol{\varphi}^T(t)\boldsymbol{\theta}$$

A general methodology applicable to a wide range of model structures.

## Examples

Find the optimal predictor,  $\hat{y}(t|t-1)$  for the following systems assuming  $Ee(t) = 0$ ,  $Ee(t)e(s) = \delta_{s,t}\lambda^2$ .

Notice that  $\hat{y}(t|t-1)$  is a function of  $\{y(s), u(s)\}_{s=-\infty}^{t-1}$ .

a)  $y(t) = e(t)$

b)  $(1 - 0.1q^{-1})y(t) = -0.5q^{-1}u(t) + e(t)$

c)  $(1 - 0.1q^{-1})y(t) = -0.5q^{-1}u(t) + (1 - 0.8q^{-1})e(t)$

## Predictions

A predictor can be described as a filter that predicts the output of a dynamic system given old measured outputs and inputs. Design the predictor by

- (i) Choosing the model structure of  $y(t)$ , *e.g.*, ARX, OE, or ARMAX.
- (ii) Choosing the predictor,  $\hat{y}(t|t-1, \boldsymbol{\theta})$ . A general predictor can be viewed as

$$\hat{y}(t|t-1, \boldsymbol{\theta}) = L_1(q^{-1}, \boldsymbol{\theta})y(t) + L_2(q^{-1}, \boldsymbol{\theta})u(t)$$

where  $L_1(q^{-1}, \boldsymbol{\theta})$  and  $L_2(q^{-1}, \boldsymbol{\theta})$  are constrained such that  $\hat{y}(t|t-1, \boldsymbol{\theta})$  depends on past data.

## Optimal Prediction

We will here consider the general model structure

$$y(t) = G(q^{-1}, \boldsymbol{\theta})u(t) + H(q^{-1}, \boldsymbol{\theta})e(t)$$

where  $E[e(t)e^T(s)] = \boldsymbol{\Lambda}(\boldsymbol{\theta})\delta_{t,s}$  and  $G(0, \boldsymbol{\theta}) = 0$ .

**Goal:** Find the optimal mean square predictor  $\hat{y}(t|t-1, \boldsymbol{\theta})$ , *i.e.*, solve

$$\min_{\hat{y}(t|t-1)} E\varepsilon(t)\varepsilon^T(t)$$

where  $\varepsilon(t) = y(t) - \hat{y}(t|t-1)$  is the prediction error, and  $\hat{y}(t|t-1)$  depends on  $\{y(s), u(s)\}_{s=-\infty}^{t-1}$ .

### Results:

Under the assumptions that

- (i)  $z(t)$  only depends on past measurements
- (ii)  $u(t)$  and  $e(s)$  are uncorrelated for  $t < s$

then

$$\hat{y}(t|t-1, \boldsymbol{\theta}) = H^{-1}(q^{-1}, \boldsymbol{\theta})G(q^{-1}, \boldsymbol{\theta})u(t) + [I - H^{-1}(q^{-1}, \boldsymbol{\theta})] y(t)$$

is the optimal mean square predictor, and  $e(t)$  the prediction error,

$$\begin{aligned}\varepsilon(t, \boldsymbol{\theta}) &= y(t) - \hat{y}(t|t-1, \boldsymbol{\theta}) \\ &= H^{-1}(q^{-1}, \boldsymbol{\theta}) [y(t) - G(q^{-1}, \boldsymbol{\theta})u(t)] \\ &= e(t)\end{aligned}$$

Hence,

$$E\varepsilon(t, \boldsymbol{\theta})\varepsilon^T(t, \boldsymbol{\theta}) = \boldsymbol{\Lambda}(\boldsymbol{\theta})$$

## Optimal Prediction for State Space Models

As an alternative to the model structure:

$$y(t) = G(q^{-1}, \boldsymbol{\theta})u(t) + H(q^{-1}, \boldsymbol{\theta})e(t),$$

it is often common to use state-space models:

$$x(t+1) = F(\boldsymbol{\theta})x(t) + B(\boldsymbol{\theta})u(t) + v(t)$$

$$y(t) = C(\boldsymbol{\theta})x(t) + e(t)$$

where  $v(t)$  and  $e(t)$  are uncorrelated white noise sequences with zero mean and covariance matrices  $R_1(\boldsymbol{\theta})$  and  $R_2(\boldsymbol{\theta})$ .

In this case the optimal mean square predictor is given by the **Kalman filter** [See “Automazione (Laboratorio)” course]

## Cost Function

How do we find the best model in the model structure?

- Minimize the prediction errors  $\varepsilon(t, \boldsymbol{\theta})$  for all  $t$ . How?
- Choose a criterion function  $V_N(\boldsymbol{\theta})$  to minimize:

$$\hat{\boldsymbol{\theta}} = \arg \min_{\boldsymbol{\theta}} V_N(\boldsymbol{\theta})$$

where  $V_N(\boldsymbol{\theta})$  depends on  $\varepsilon(t, \boldsymbol{\theta})$  in a suitable manner.

Depending on the choice of model structure, predictor filters and criterion function, the minimization of the loss function is more or less difficult.

For single-output systems the following criterion function is most often used

$$V_N(\boldsymbol{\theta}) = \frac{1}{N} \sum_{t=1}^N \varepsilon^2(t, \boldsymbol{\theta})$$

In general, the cost function is chosen as

$$V_N(\boldsymbol{\theta}) = h(\mathbf{R}_N(\boldsymbol{\theta}))$$

where  $h(\cdot)$  is a scalar-valued monotonically increasing function, and  $\mathbf{R}_N(\boldsymbol{\theta})$  is the sample covariance matrix of the prediction errors,

$$\mathbf{R}_N(\boldsymbol{\theta}) = \frac{1}{N} \sum_{t=1}^N \varepsilon(t, \boldsymbol{\theta}) \varepsilon^T(t, \boldsymbol{\theta}).$$

**Ex:**  $h(\cdot) = \text{tr}(\cdot)$ , or  $h(\cdot) = \det(\cdot)$ .

### A PEM Algorithm

To define a PEM the user has to make the following choices:

- Choice of model structure. How should  $G(q^{-1}, \boldsymbol{\theta})$ ,  $H(q^{-1}, \boldsymbol{\theta})$  and  $\Lambda(\boldsymbol{\theta})$  be parameterized?
- Choice of predictor  $\hat{y}(t|t-1, \boldsymbol{\theta})$ . Usually the optimal mean square predictor is used.
- Choice of criterion function  $V(\boldsymbol{\theta})$ . A scalar-valued function of all the prediction errors  $\varepsilon(1, \boldsymbol{\theta}), \dots, \varepsilon(N, \boldsymbol{\theta})$ , which will assess the performance of the predictor used.

### I. Analytical solution exists

If the predictor is a linear function of the unknown parameters,

$$\hat{y}(t|t-1, \boldsymbol{\theta}) = \boldsymbol{\varphi}^T(t)\boldsymbol{\theta},$$

and the criterion function  $V_N(\boldsymbol{\theta})$  is simple enough, a closed form solution can be found. For example, when

$$V_N(\boldsymbol{\theta}) = \frac{1}{N} \sum_{t=1}^N \varepsilon^2(t, \boldsymbol{\theta}) = \frac{1}{N} \sum_{t=1}^N (y(t) - \boldsymbol{\varphi}^T(t)\boldsymbol{\theta})^2,$$

it is clear that the PEM is equivalent to linear regression (the least squares method). This holds for example for ARX or FIR models but **not** for ARMAX and OE models.

### II. No analytical solution exists

For general criterion functions, and predictors that depend non-linearly on the data, a numerical search algorithm is required to find the  $\boldsymbol{\theta}$  that minimizes  $V_N(\boldsymbol{\theta})$ .

Numerical minimization:

- Nonlinear  $\Rightarrow$  local minima may exist.
- Time consuming (convergence rate) and computationally complex.
- Initialization.



Different (standard) methods available:

- The **Newton-Raphson** algorithm

$$\hat{\boldsymbol{\theta}}^{(k+1)} = \hat{\boldsymbol{\theta}}^{(k)} - \alpha_k [V''(\hat{\boldsymbol{\theta}}^{(k)})]^{-1} V'(\hat{\boldsymbol{\theta}}^{(k)})$$

The derivatives of the loss function can be computationally complex to evaluate. Fast convergence.

- The **Gauss-Newton** algorithm is a computationally less intensive algorithm with a theoretically lower rate of convergence which can be used as an alternative.
- **Gradient based** methods are simpler to apply, but has a slow convergence rate.
- **Grid search.** Search the whole parameter space. VERY time consuming.

## Theoretical Analysis

Assumptions

- The data  $\{u(t), y(t)\}$  are stationary processes.
- The input is persistently exciting.
- $V_N''(\boldsymbol{\theta})$  is nonsingular around the minimum points of  $V_N(\boldsymbol{\theta})$ .
- The filters  $G(q^{-1}, \boldsymbol{\theta})$  and  $H(q^{-1}, \boldsymbol{\theta})$  are smooth differentiable functions of the parameter vector.

What happens with the estimate  $\hat{\boldsymbol{\theta}}_N$  as  $N \rightarrow \infty$  ?

## Consistency:

$$\hat{\boldsymbol{\theta}}_{\infty} \triangleq \lim_{N \rightarrow \infty} \hat{\boldsymbol{\theta}}_N = \arg \min_{\boldsymbol{\theta}} V_{\infty}(\boldsymbol{\theta})$$

$$V_{\infty}(\boldsymbol{\theta}) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=1}^N \varepsilon^2(t, \boldsymbol{\theta}) = E \varepsilon^2(t, \boldsymbol{\theta})$$

The PEM estimates are robust and consistent:

- As  $N \rightarrow \infty$ ,  $\hat{\boldsymbol{\theta}}_N$  converges to a minimum point of  $V_{\infty}(\boldsymbol{\theta})$ .
- If the model structure includes the true system ( $D_T$  non-empty) then the PEM is system identifiable ( $\hat{\boldsymbol{\theta}}_{\infty} \in D_T$ ).
- If there is a unique vector  $\boldsymbol{\theta}_0$  that gives an exact description of the system ( $D_T$  contains one point), then the system is parameter identifiable. The PEM estimate is **consistent** ( $\hat{\boldsymbol{\theta}}_N \rightarrow \boldsymbol{\theta}_0$  as  $N \rightarrow \infty$ ).

**Asymptotic distribution:** Asymptotic distribution of the parameter estimates (assuming that the model is parameter identifiable,  $\hat{\boldsymbol{\theta}}_N \rightarrow \boldsymbol{\theta}_0$ )

- The parameter estimation errors are asymptotically Gaussian distributed with zero mean and variance  $\mathbf{P}$

$$\sqrt{N}(\hat{\boldsymbol{\theta}}_N - \boldsymbol{\theta}_0) \rightarrow N(0, \mathbf{P})$$

- For single-output systems the covariance matrix of the parameter estimates are given by

$$\mathbf{P} = \boldsymbol{\Lambda} [E \Psi(t, \boldsymbol{\theta}_0) \Psi^T(t, \boldsymbol{\theta}_0)]^{-1}$$

where

$$\Psi(t, \boldsymbol{\theta}) = - \left( \frac{\partial \varepsilon(t, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right)^T$$

and  $E e(t) e^T(t) = \boldsymbol{\Lambda}$ .

## Accuracy of linear regression for static/dynamic case

### Static case ( $N$ finite)

- $\hat{\boldsymbol{\theta}}$  unbiased.
- $\sqrt{N}(\hat{\boldsymbol{\theta}}_N - \boldsymbol{\theta}_0)$  is Gaussian distributed  $N(0, P)$ ,

$$P = \Lambda \left( \frac{1}{N} \sum_{t=1}^N \boldsymbol{\varphi}(t, \boldsymbol{\theta}_0) \boldsymbol{\varphi}^T(t, \boldsymbol{\theta}_0) \right)^{-1}$$

### Dynamic case (for $N \rightarrow \infty$ ).

- $\hat{\boldsymbol{\theta}}$  is consistent.
- $\sqrt{N}(\hat{\boldsymbol{\theta}}_N - \boldsymbol{\theta}_0)$  is asymptotically Gaussian distributed  $N(0, P)$ .

$$P = \Lambda [E\boldsymbol{\varphi}(t, \boldsymbol{\theta}_0) \boldsymbol{\varphi}^T(t, \boldsymbol{\theta}_0)]^{-1}$$

## Statistical efficiency

- A method is said to be statistically efficient if its estimates have the smallest possible variance.
- The smallest possible variance of any (asymptotically) unbiased estimator is given by the Cramér-Rao lower bound.
- For Gaussian disturbances the PEM method is statistically efficient (equivalent to the maximum likelihood (ML) method) if
  - Single-output:  $V_N(\boldsymbol{\theta}) = \frac{1}{N} \sum_{t=1}^N \varepsilon^2(t, \boldsymbol{\theta})$ .
  - Multi-output:  $V_N(\boldsymbol{\theta}) = \text{tr}(S R_N(\boldsymbol{\theta}))$  and  $S = \Lambda^{-1}(\boldsymbol{\theta}_0)$ , or  $V_N(\boldsymbol{\theta}) = \det(R_N(\boldsymbol{\theta}))$ .

## Approximation

The true system is often more complex than the model structure (under-parametrization,  $D_T$  is empty).

- Still,  $\hat{\boldsymbol{\theta}}_N$  converges to a minimum point of  $V_\infty(\boldsymbol{\theta})$  as  $N \rightarrow \infty$ .
- We cannot expect  $G(q^{-1}, \boldsymbol{\theta}) \equiv G_0(q^{-1})$  and  $H(q^{-1}, \boldsymbol{\theta}) \equiv H_0(q^{-1})$  to hold.
- The model-fit can be controlled by pre-filtering the data,

$$u_F(t) = F(q^{-1})u(t), \quad y_F(t) = F(q^{-1})y(t),$$

or by choosing an appropriate input.

- The OE model structure is useful.

## Conclusions

- The PEM is a general method to obtain a parametric model of a dynamic system. The following choices define a prediction error method:
  - Choice of model structure;
  - choice of predictor;
  - choice of criterion function.
- The PEM principle is to minimize the prediction errors given a certain model structure and predictor.
- The PEM principle leads to parameter estimates that have several nice properties (in general, consistent and statistically efficient estimates).

- Approximation. The PEM is useful also for under-parameterized models. The model-fit can be controlled by pre-filtering the data, or by choosing an appropriate input.
- If the prediction errors depend linearly on the parameter vector the PEM estimates are obtained through linear regression (e.g., ARX and FIR models).
- In the case of more complicated model structures a nonlinear search algorithm is required to obtain the PEM estimates (e.g., ARMAX, OE, etc.).