# Non–linear system modelling from noisy data & FDI using multiple–model approach

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## Summary

- **Hybrid system modelling and identification**
  - ⇒ Complex process descriptions
  - ⇒ LS, fuzzy data clustering, NNs, etc.
  - ⇒ Continuous piecewise affine (PWL) models from noisy data
- **System interpolation properties**
  - ⇒ target function & its first order derivative interpolation
- **Noise identification & rejection problem: R. E. Kalman**
  - ⇒ dynamic linear system identification

## Non-linear system modelling

- $y(t+n) = F\big(y(t+n-1), \cdots, y(t), u(t+n-1), \cdots, u(t)\big), \quad t = 0, 1, \ldots$
  - $\Rightarrow$ $u(\cdot) \in \mathcal{U}$, $y(\cdot) \in \mathcal{Y}$
  - $\Rightarrow$ continuous non–linear function $F(\cdot)$: $\mathcal{A}_n \rightarrow \mathcal{Y}$
  - $\Rightarrow$ $\mathcal{A}_n \in \mathcal{U}^n \times \mathcal{Y}^n$
  - $\Rightarrow$ finite system memory $n$

- **Non-linear system identification $\hookrightarrow$ piecewise model approximation**
  - $\Rightarrow$ Interpolation properties and characteristics





## Submodel structure

- **Parametric submodel collection**
  - $\Rightarrow$ $y(t+n) = \sum_{j=0}^{n-1} \alpha_j^{(i)} y(t+j) + \sum_{j=0}^{n-1} \beta_j^{(i)} u(t+j) + b^{(i)}, \quad t = 0, 1, \ldots$
  - $\Rightarrow$ $\mathbf{x}_n(t) = [y(t), \cdots, y(t+n-1), u(t), \cdots, u(t+n-1)]^T$

- **Switching function**
  - $\Rightarrow$ $\chi_i\big(\mathbf{x}_n(t)\big) = \begin{cases} \chi_i\big(\mathbf{x}_n(t)\big) = 1 \text{ if } \mathbf{x}_n(t) \in \mathcal{A}_n^{(i)} \\ \chi_i\big(\mathbf{x}_n(t)\big) = 0 \text{ otherwise} \end{cases}$
  - $\Rightarrow$ $\{\mathcal{A}_n^{(1)}, \ldots, \mathcal{A}_n^{(M)}\}$ is a partition of $\mathcal{A}_n$

## Model structure

- **Piecewise affine system**

  $\Rightarrow \; y(t+n) = f\big(\mathbf{x}_n(t)\big) = \sum_{i=1}^{M} \chi_i\big(\mathbf{x}_n(t)\big)\, [\mathbf{x}_n(t), 1]^T\, \mathbf{a}_n^{(i)}$

- **Submodel parameters**

  $\Rightarrow \; \mathbf{a}_n^{(i)} = \left[\alpha_0^{(i)}, \ldots, \alpha_{n-1}^{(i)}, \beta_0^{(i)}, \ldots, \beta_{n-1}^{(i)}, b^{(i)}\right]^T$

  $\Rightarrow$ The model is affine in each $\mathcal{A}_n^{(i)}$





## Affine model approximation capabilities

- **Piecewise linear/affine functions $f(\cdot)$ can interpolate an arbitrary non–linear model $F(\cdot)$**

  $\Rightarrow \; {}^{1}\|F - f\|_w < \epsilon$

- **Function and its first derivative approximation**

  $\Rightarrow \; \max_{\mathbf{x}_n \in \gamma} |F(\mathbf{x}_n) - f(\mathbf{x}_n)| \leq \epsilon$

  $\Rightarrow \; \max_{\mathbf{x}_n \in \gamma} \left| \frac{\partial F}{\partial x_n^j}(\mathbf{x}_n) - \frac{\partial f}{\partial x_n^j}(\mathbf{x}_n) \right| \leq \epsilon, \quad j = 1, \ldots, 2n.$

## Model continuity

- **$F(\cdot)$ is continuous $\hookrightarrow$ $f(\cdot)$ has to be continuous in $\mathcal{A}_n$**
- **Parameter constraints**

$$\Rightarrow \lim_{\substack{\mathbf{x}_n(t)\to\bar{\mathbf{x}}_n \\ \mathbf{x}_n(t)\in\mathcal{A}_n^{(i')}}} f\big(\mathbf{x}_n(t)\big) = \lim_{\substack{\mathbf{x}_n(t)\to\bar{\mathbf{x}}_n \\ \mathbf{x}_n(t)\in\mathcal{A}_n^{(i'')}}} f\big(\mathbf{x}_n(t)\big)$$

$$\Rightarrow \bar{\mathbf{x}}_n \in \mathcal{A}_n^{(i')} \cap \mathcal{A}_n^{(i'')}$$

$$\Rightarrow \bar{\mathbf{x}}_n(t)^T\,\mathbf{a}_n^{(i')} = \bar{\mathbf{x}}_n(t)^T\,\mathbf{a}_n^{(i'')}$$

- **Infinite number of constraints!**

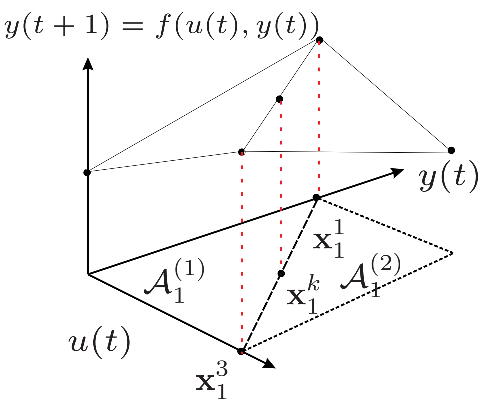
$\Rightarrow$ Theorem $\hookrightarrow$ $\mathcal{A}_n^{(i)}$ should be convex polyhedra

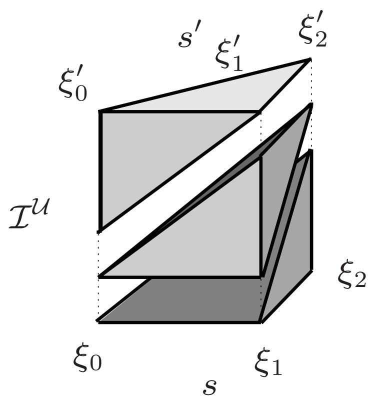




## Continuity example

- **An example of a partition of a 2–dimensional space $\mathcal{A}_1 = \mathcal{U} \times \mathcal{Y}$**




$\Rightarrow$ Regions $\mathcal{A}_1^{(1)}$ and $\mathcal{A}_1^{(2)}$ have triangular shape

# Continuity constraints

- **"Triangulation" of $\mathcal{A}_n$**
  - ⇒ $\mathcal{A}_n$ partition into $2n$–dimensional simplexes
  - ⇒ disjoint simplexes or a whole $k$–dimensional boundary in common with $k = 0, 1, \ldots, 2n-1$.
  - ⇒ Constraints in $2n+1$ independent vertices of each simplex
- **Continuity constraint matrix $C_n$**
  - ⇒ $C_n A_n = \mathbf{0}$
  - ⇒ $A_n = \left[ \begin{array}{ccc} \mathbf{a}_n^{(1)} & \ldots & \mathbf{a}_n^{(M)} \end{array} \right]$





# Local affine model identification: ideal case

- $\{u(t), y(t)\}_{t=0}^{L_i} \in \mathcal{A}_n^{(i)} \hookrightarrow$ **piecewise affine system noiseless data**
  - ⇒ $n$ and $\mathbf{a}_n^{(i)}$ identification in $\mathcal{A}_n^{(i)}$
  - ⇒ $X_k^{(i)} = \left[ \begin{array}{ccc} y(k) & \mathbf{x}_k^T(0) & 1 \\ y(k+1) & \mathbf{x}_k^T(1) & 1 \\ \vdots & \vdots & \\ y(k+N_i-1) & \mathbf{x}_k^T(N_i-1) & 1 \end{array} \right], \quad \Sigma_k^{(i)} = \left(X_k^{(i)}\right)^T X_k^{(i)}$
- **$n$ and $\mathbf{a}_n^{(i)}$ estimates in $\mathcal{A}_n^{(i)}$**
  - ⇒ $\Sigma_2^{(i)}, \ \Sigma_3^{(i)}, \ \ldots \ \Sigma_k^{(i)}, \ \ldots$ increasing sequence
  - ⇒ $\Sigma_k^{(i)} \geq \mathbf{0} \hookrightarrow$ model order $n = k \hookrightarrow \Sigma_n^{(i)} \mathbf{a}_n^{(i)} = \mathbf{0}$

## Local affine model identification: noisy case

- $\{u(t), y(t)\}_{t=0}^{L_i} \in \mathcal{A}_n^{(i)} \hookrightarrow$ **piecewise affine system noisy data**

  $\Rightarrow$ Additive measurement noise $\hookrightarrow$ EIV model

- $\begin{cases} u(t) & = & u^*(t) + \tilde{u}(t) \\ y(t) & = & y^*(t) + \tilde{y}(t) \end{cases}$

  $\Rightarrow$ $\{\tilde{u}(t), \tilde{y}(t)\}$ are region independent

  $\Rightarrow$ white $\{\tilde{u}(t), \tilde{y}(t)\}$ are independent of $\{u^*(t), y^*(t)\}$

- $\bar{\tilde{\sigma}}_u$, $\bar{\tilde{\sigma}}_y$ **input and output noise variances**

  $\Rightarrow$ $\Sigma_k^{(i)}$ s.t. $\Sigma_k^{(i)} = \Sigma_k^{*(i)} + \bar{\tilde{\Sigma}}_k$ with $\bar{\tilde{\Sigma}}_k = \text{diag}[\bar{\tilde{\sigma}}_y I_{k+1}, \bar{\tilde{\sigma}}_u I_k, 0] \geq 0$





## Local affine model identification: real case

- $n$ **and** $\mathrm{a}_n^{(i)}$ **estimates in** $\mathcal{A}_n^{(i)}$

  $\Rightarrow$ analysis of $\Sigma_2^{(i)}$, $\Sigma_3^{(i)}$, ... $\Sigma_k^{(i)}$, ...

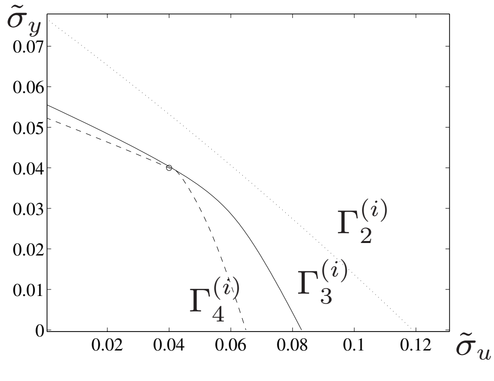
- $\tilde{\Sigma}_k$ **identification**

  $\Rightarrow$ computation of $\Sigma_k^{*(i)} = \Sigma_k^{(i)} - \tilde{\Sigma}_k \geq 0$

  $\Rightarrow$ $(\tilde{\sigma}_u, \tilde{\sigma}_y)$ s.t. $\tilde{\Sigma}_k = \text{diag}[\tilde{\sigma}_y I_{k+1}, \tilde{\sigma}_u I_k, 0]$

- **Infinite solutions for any** $k$**: curve** $\Gamma_k^{(i)}(\tilde{\sigma}_y, \tilde{\sigma}_u) = 0$

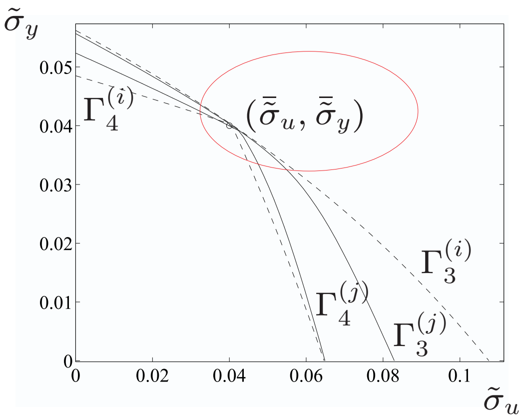
  $\Rightarrow$ curve geometrical properties $\hookrightarrow$ $n$ estimate




*Singularity curves for $n = 3$*

# Multiple–model identification: ideal case

- **Common noise characteristics in all $\mathcal{A}_n^{(i)}$**
  - $\Rightarrow$ model structure and $\mathcal{A}_n$ partition noise independent
  - $\Rightarrow$ singularity surface common point $(\bar{\tilde{\sigma}}_y, \bar{\tilde{\sigma}}_u)$
- **$n$ and $\mathbf{a}_n^{(i)}$ identification** $(i = 1, \ldots, M)$
  - $\Rightarrow$ $(\Sigma_n^{(i)} - \bar{\tilde{\Sigma}}_n)\mathbf{a}_n^{(i)} = \mathbf{0}$ for $i = 1, \ldots, M$

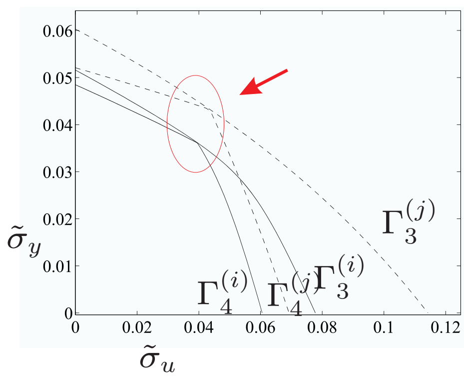






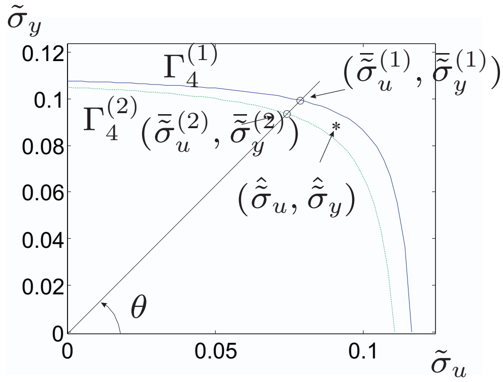

# Multiple–model identification: real applications

- **No common point among curves $\Gamma_n^{(i)} = 0$**
  - $\Rightarrow$ approximation of the ideal case
- $\Gamma_{n+1}^{(i)} = 0$ **intersects** $\Gamma_n^{(i)} = 0$
  - $\Rightarrow$ $\Sigma_{n+1}^{*(i)}$ double singularity
- $n \hookrightarrow (\tilde{\sigma}_u, \tilde{\sigma}_y) \in \Gamma_{n+1}^{(i)} = 0$ **s.t.** $\Sigma_{n+1}^{*(i)}$ **double singular**
  - $\Rightarrow$ $k \hookrightarrow \max_{i=1,\ldots,M} \lambda_k^{(i)} < \epsilon$
- $(\bar{\tilde{\sigma}}_u^{(i)}, \bar{\tilde{\sigma}}_y^{(i)})$ **s.t.** $\Sigma_n^{*(i)} = \Sigma_n^{(i)} - \tilde{\Sigma}_n^{(i)} \geq 0$
  - $\Rightarrow$ $\tilde{\Sigma}_n^{(i)} = \mathrm{diag}[\bar{\tilde{\sigma}}_u^{(i)} I_{n+1}, \bar{\tilde{\sigma}}_y^{(i)} I_n, 0]$



## Real noise identification

- $(\bar{\tilde{\sigma}}_u^{(i)}, \bar{\tilde{\sigma}}_y^{(i)})$ **computation** $\hookrightarrow$ **optimisation problem**
  - $\Rightarrow$ $(\bar{\tilde{\sigma}}_u^{(i)}, \bar{\tilde{\sigma}}_y^{(i)})$ and $(\bar{\tilde{\sigma}}_u^{(j)}\ \bar{\tilde{\sigma}}_y^{(j)})$ distance minimisation
  - $\Rightarrow$ continuity constraints satisfaction
- **Cost function** $\hookrightarrow$ $J((\bar{\tilde{\sigma}}_u^{(1)}, \bar{\tilde{\sigma}}_y^{(1)}), \ldots, (\bar{\tilde{\sigma}}_u^{(M)}, \bar{\tilde{\sigma}}_y^{(M)})) = d\left((\bar{\tilde{\sigma}}_u^{(1)}, \bar{\tilde{\sigma}}_y^{(1)}), \ldots, (\bar{\tilde{\sigma}}_u^{(M)}, \bar{\tilde{\sigma}}_y^{(M)})\right) + (C_n A_n)^T\, H C_n A_n$
- **Distance** $\hookrightarrow$ $d\left((\bar{\tilde{\sigma}}_u^{(1)}, \bar{\tilde{\sigma}}_y^{(1)}), \ldots, (\bar{\tilde{\sigma}}_u^{(M)}, \bar{\tilde{\sigma}}_y^{(M)})\right) = \sum_{i=1}^{M} \sum_{j=i+1}^{M} \sqrt{(\bar{\tilde{\sigma}}_u^{(i)} - \bar{\tilde{\sigma}}_u^{(j)})^2 + (\bar{\tilde{\sigma}}_y^{(i)} - \bar{\tilde{\sigma}}_y^{(j)})^2}$



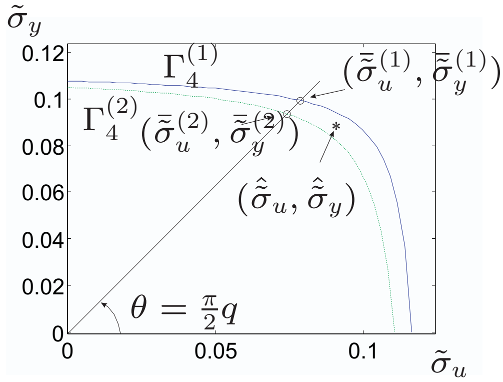

*Minimisation problem*





## Noise identification enhancement

- **Parametrisation of** $\Gamma_n^{(i)}(\bar{\tilde{\sigma}}_u^{(i)}, \bar{\tilde{\sigma}}_y^{(i)}) = 0$
  - $\Rightarrow$ complexity reduction: $2M$ independent variables
  - $\Rightarrow$ $\begin{cases} \bar{\tilde{\sigma}}_u^{(i)} = \rho^{(i)} \cos \frac{\pi}{2} q \\ \bar{\tilde{\sigma}}_y^{(i)} = \rho^{(i)} \sin \frac{\pi}{2} q \end{cases}$ (polar coordinates)
  - $\Rightarrow$ $\rho^{(i)}$ s.t. $\Gamma_n^{(i)}(\rho^{(i)} \cos \frac{\pi}{2} q, \rho^{(i)} \sin \frac{\pi}{2} q) = 0$ with $q \in [0, 1]$
- $J(q) = d\left((\bar{\tilde{\sigma}}_u^{(1)}(q), \bar{\tilde{\sigma}}_y^{(1)}(q)), \ldots, (\bar{\tilde{\sigma}}_u^{(M)}(q), \bar{\tilde{\sigma}}_y^{(M)}(q))\right) + (C_n A_n)^T\, H C_n A_n$ $\hookrightarrow$ **new cost function**
  - $\Rightarrow$ $(\Sigma_n^{(i)} - \tilde{\Sigma}_n^{(i)})\mathbf{a}_n^{(i)} = \mathbf{0}$ for $i = 1, \ldots, M$




*Minimisation problem*

## Simulation of a simple multiple–model

- $y(t+1) = \begin{cases} 0.5u(t) - 0.5y(t) + 0.5 & \textbf{if } \mathbf{x}(t) \in \mathcal{A}_1^{(1)} \\ -0.5u(t) + 0.5y(t) + 0.5 & \textbf{if } \mathbf{x}(t) \in \mathcal{A}_1^{(2)} \end{cases}$

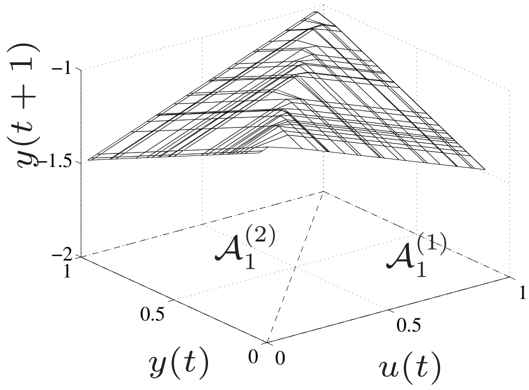
  ⇒ continuous piecewise affine dynamic model identification

- $\mathcal{A}_n$: $\{u(t), y(t)\} \in [0,1] \times [0,1]$

  ⇒ $M = 2$ two triangular regions $\mathcal{A}_1^{(1)}$ and $\mathcal{A}_1^{(2)}$

  ⇒ $L = 500$ data and $\sigma_u = \sigma_y = 0.3$

- **Evaluation test**



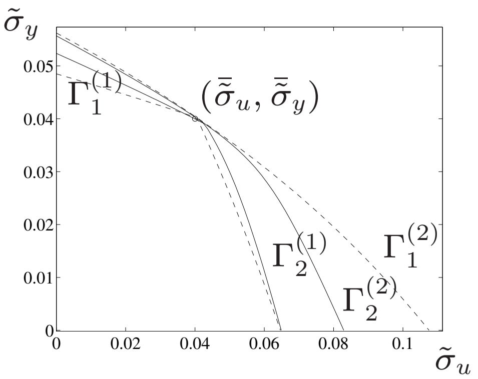

The target function

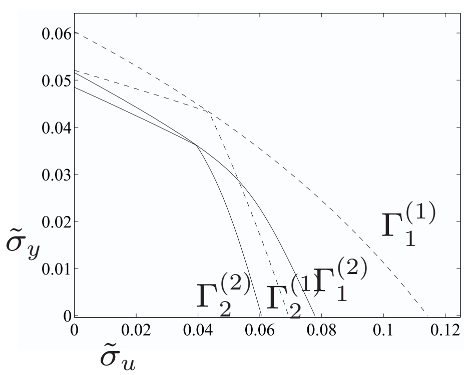




## Piecewise affine model identification: noise

- $\Gamma_k^{(i)} = 0$ **(**$i, k = 1, 2$**) for** $\mathcal{A}_1^{(1)}$ **and** $\mathcal{A}_1^{(2)}$

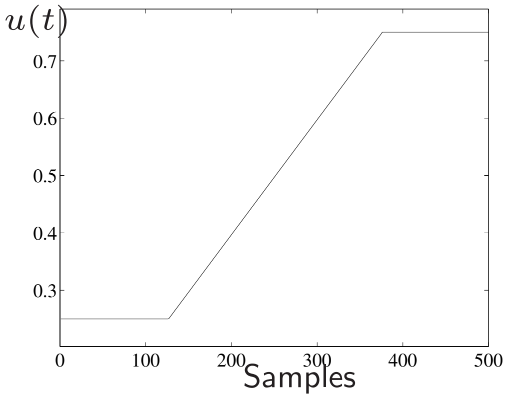


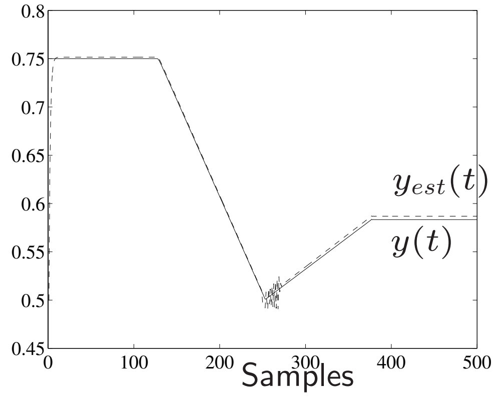

⇒ Ideal case: $(\bar{\tilde{\sigma}}_u, \bar{\tilde{\sigma}}_y) = (0.09, 0.09)$

⇒ Real case: $(\bar{\tilde{\sigma}}_u^{(1)}, \bar{\tilde{\sigma}}_y^{(1)}) = (0.080, 0.095)$ and $(\bar{\tilde{\sigma}}_u^{(2)}, \bar{\tilde{\sigma}}_y^{(2)}) = (0.075, 0.1)$

# Piecewise dynamic model simulation




(e) Input signal




(f) Model outputs

⇒ ideal case: $10^{-5}\%$ parameter identification accuracy

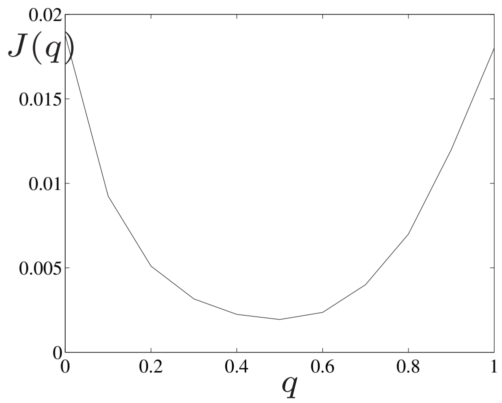
⇒ real case: $15\%$ parameter estimation precision

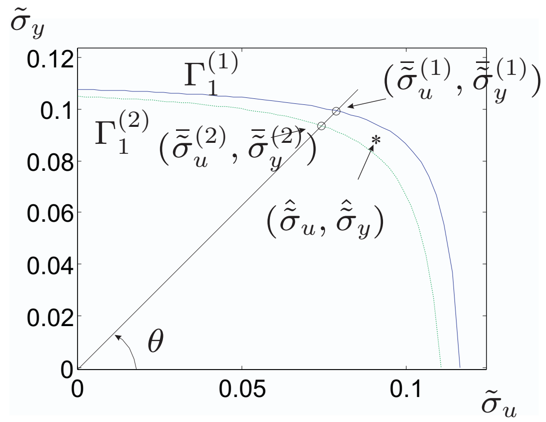




# Optimisation problem results

$n$: $\max_{i=1,2} \lambda_1^{(i)} = 2.8082 \times 10^{-4}$ **and** $\max_{i=1,2} \lambda_2^{(i)} = 2.3007 \times 10^{-4} \hookrightarrow n = 1$




⇒ cost function $J_{\min}(q)$ for $q \cong 0.5$ with $\mathrm{H} = diag[1, 1]$

# Multiple–model approach to FDI: system modelling

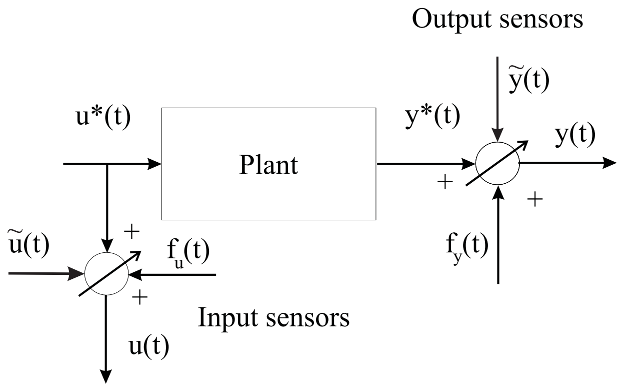
- **FDI using $u(t)$ and $y(t)$**

$\Rightarrow$ measurements

$$\begin{cases} u(t) &= u^*(t) + \tilde{u}(t) + f_u(t) \\ y(t) &= y^*(t) + \tilde{y}(t) + f_y(t) \end{cases}$$

$\Rightarrow$ $\tilde{u}(t)$ and $\tilde{y}(t)$ sensor noises

$\Rightarrow$ $f_u(t)$ and $f_y(t)$ fault signals

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# Residual generation

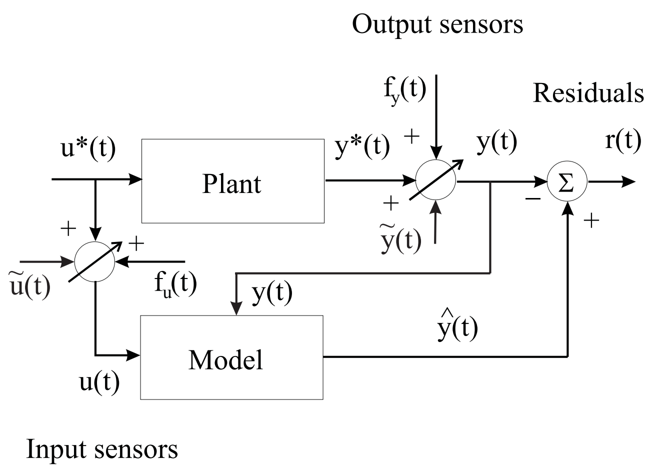
- **Symptoms generation**

$\Rightarrow$ $r(t) = \hat{y}(t) - y(t)$

- **Symptoms evaluation**

$$\Rightarrow \quad |r(t)| \begin{cases} \leq \text{Threshold} & \text{No fault,} \\ > \text{Threshold} & \text{Fault.} \end{cases}$$

$\Rightarrow$ threshold logic

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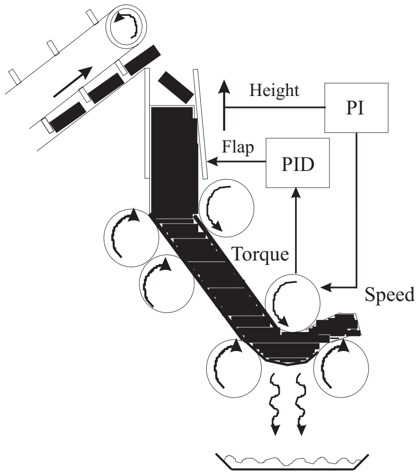
# Application example

- **Identification and FDI of a sugar cane crushing mill**
  - ⇒ Inputs: turbine speed and chute flap $r = 2$
  - ⇒ Outputs: turbine torque and chute height $m = 2$




- **Samples from normal operating records**
- **Data clustering and PWL identification**





# Process identification

- **PWL Toolbox for Matlab®**
  - ⇒ domain partition, data clustering and simulations
  - ⇒ toolbox modification $\hookrightarrow$ Frisch scheme dynamic identification
  - ⇒ $M = 2$ clusters $m = 2$ outputs and $r = 2$ inputs
- **Input–output data pre-filtering**
  - ⇒ $n = 2$ model order for each output
- **Model validation**
  - ⇒ $J_1 = 0.0084$ $(VAF = 99\%)$ and $J_2 = 0.0440$ $(VAF = 90\%)$

# Input-output sensor FDI

- **Input and output sensor fault simulation, $u(t)$ and $y(t)$**
  - ⇒ step and ramp additive faults, $f_u(t)$ and $f_y(t)$
  - ⇒ transient condition fault occurrence
- **Fault free and faulty residual analysis**

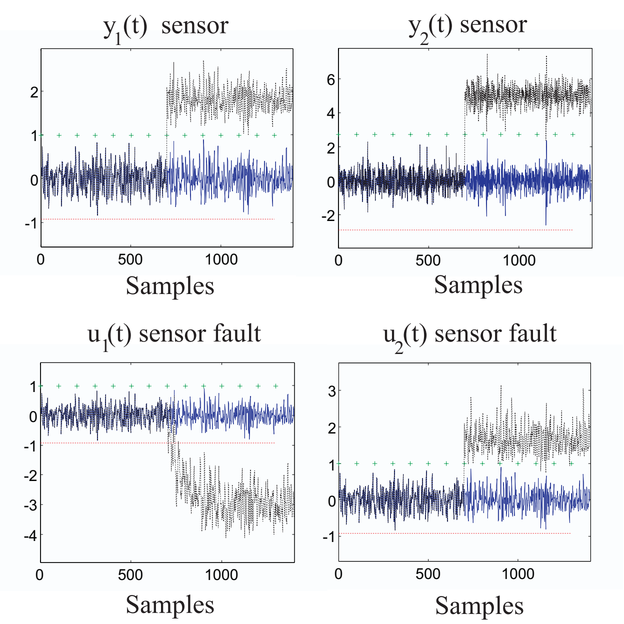
| Sensor | $u_1(t)$ | $u_2(t)$ | $y_1(t)$ | $y_2(t)$ |
|---|---|---|---|---|
| Fault | 2.0% | 2.5% | 1.0% | 7.0% |

⇒ per cent of the signal mean values





# Fault–free and faulty residual examples



# Conclusions

- **Non–linear model identification from noisy data**
- **Piecewise affine models as universal approximators**
  - $\Rightarrow$ model structure estimate $\hookrightarrow$ Frisch scheme method
  - $\Rightarrow$ data noise rejection $\hookrightarrow$ R.E. Kalman
- **Continuity constraint fulfilment and region-independent noise assumptions**
- **Further investigations**
  - $\Rightarrow$ Optimal domain partition
  - $\Rightarrow$ Observers/filters for PWL models & EA problem





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